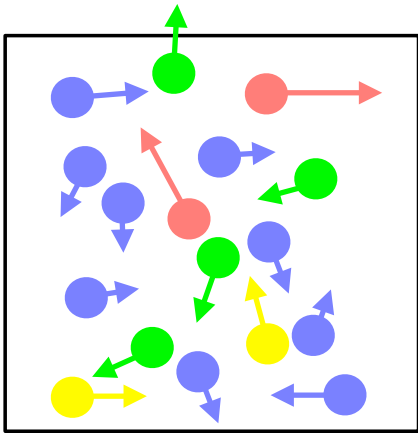


Loading non-Maxwellian velocity distributions in particle-in-cell (PIC) simulation

S. Zenitani (IWF), S. Usami (NIFS),
& S. Matsukiyo (Kyushu U)

AGU Annual Meeting 2025

Velocity distribution in PIC simulation



- Maxwell-Boltzmann distribution

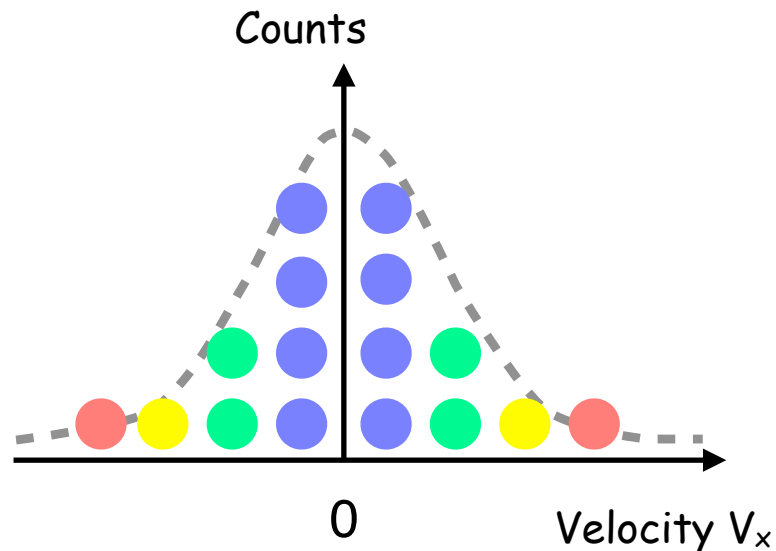
$$f(\mathbf{v})d^3v = N\left(\frac{m}{2\pi T}\right)^{\frac{3}{2}} \exp\left(-\frac{mv^2}{2T}\right)d^3v$$

- Box-Muller method (Box & Muller 1958)

- Two uniform random variates: $U_1, U_2 \in (0,1)$

$$n_1 = \sqrt{-2 \ln U_1} \cos(2\pi U_2)$$

$$n_2 = \sqrt{-2 \ln U_1} \sin(2\pi U_2)$$



How to start from
non-Maxwellian velocity
distributions?

Monte-Carlo methods for VDFs

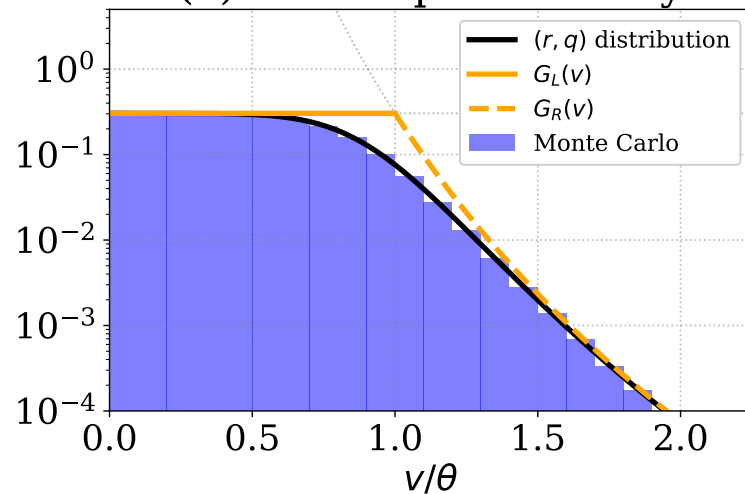
- Kappa distributions (SZ & Nakano 2022 PoP)
- Loss-cone distributions (SZ & Nakano 2023 JGR)
- Nine distributions (This work; [arXiv:2510.11890](https://arxiv.org/abs/2510.11890))
 - 1. (r,q) distribution
 - 2. Regularized Kappa distribution
 - 3. Subtracted Kappa distribution
 - 4 & 5. Ring and shell distributions
 - 6 & 7. Ring and shell Maxwellians
 - 8. Super Gaussian distribution
 - 9. Filled-shell distribution

1. (r,q) distribution

[Zaheer+ 2004, Qureshi+ 2004]

$$f_{\text{rq}}(\mathbf{v}) d^3 v = N \cdot C_{\text{rq}} \left(1 + \frac{1}{q-1} \left(\frac{v_{\parallel}^2}{\theta_{\parallel}^2} + \frac{v_{\perp}^2}{\theta_{\perp}^2} \right)^{1+r} \right)^{-q} d^3 v,$$
$$C_{\text{rq}} = \frac{3\Gamma(q)}{4\pi(q-1)^{\frac{3}{2(1+r)}} \theta_{\parallel} \theta_{\perp}^2 \Gamma\left(1 + \frac{3}{2(1+r)}\right) \Gamma\left(q - \frac{3}{2(1+r)}\right)},$$

(a) Phase-space density



- Generalization of
 - Kappa & flattop VDFs
- Two methods
 - (A) Piecewise rejection method
 - (B) Beta-prime method

Beta-prime method

- Generalized beta-prime distribution

$$B'(x; \alpha, \beta, p, q) = \frac{\alpha p \Gamma(\alpha + \beta)}{q^{\alpha p} \Gamma(1 + \alpha) \Gamma(\beta)} \left(1 + \left(\frac{x}{q}\right)^p\right)^{-(\alpha + \beta)} x^{\alpha p - 1}$$

- (r,q) distribution in spherical coordinates

$$f_{\text{rq}}(v) 4\pi v^2 = N_0 B' \left(v; \frac{3}{2(1+r)}, q - \frac{3}{2(1+r)}, 2(1+r), (q-1)^{\frac{1}{2(1+r)}} \theta \right)$$

- Beta-prime random number

$$X_{B'(\alpha, \beta, p, q)} = q \left(\frac{X_{\text{Ga}(\alpha, \delta)}}{X_{\text{Ga}(\beta, \delta)}} \right)^{1/p}$$

Beta-prime random number

Gamma random numbers

Algorithm 3-1: Beta-prime method

```

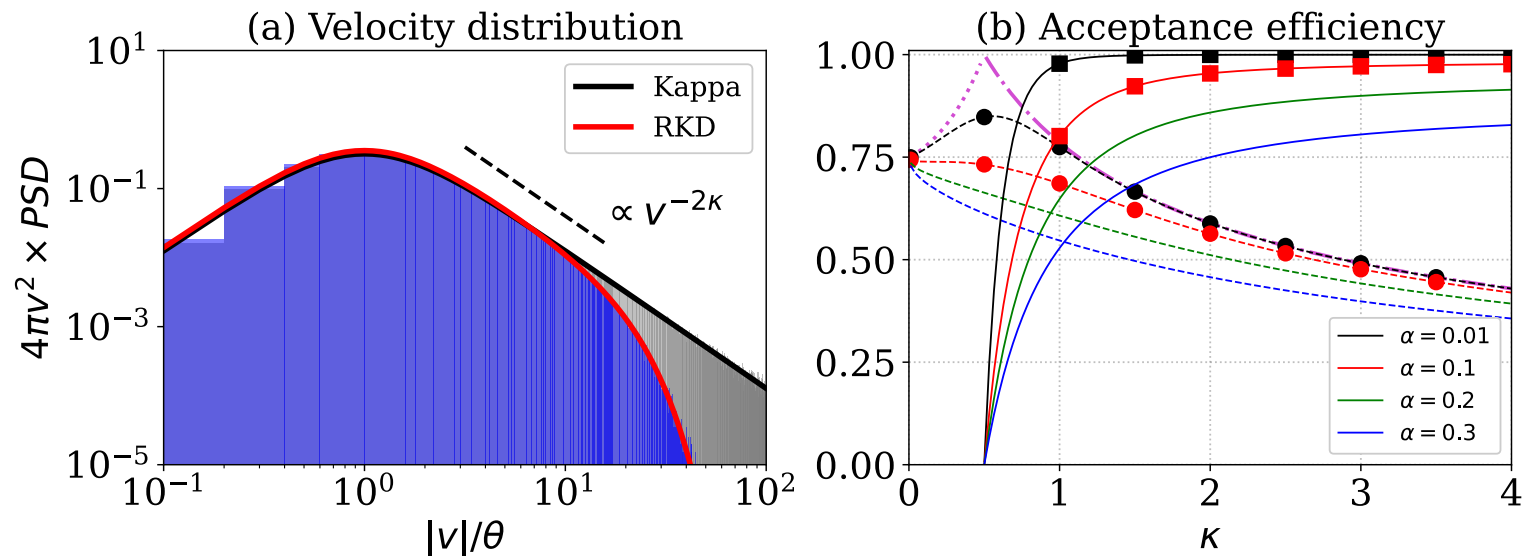
generate  $X_1 \sim \text{Ga}(\frac{3}{2(1+r)}, 1)$ 
generate  $X_2 \sim \text{Ga}(q - \frac{3}{2(1+r)}, 1)$ 
generate  $U_1, U_2 \sim U(0, 1)$ 
 $x \leftarrow [(q-1)X_1/X_2]^{\frac{1}{2(1+r)}}$ 
 $v_{\parallel} \leftarrow \theta_{\parallel} x (2U_1 - 1)$ 
 $v_{\perp 1} \leftarrow 2\theta_{\perp} x \sqrt{U_1(1-U_1)} \cos(2\pi U_2)$ 
 $v_{\perp 2} \leftarrow 2\theta_{\perp} x \sqrt{U_1(1-U_1)} \sin(2\pi U_2)$ 
return  $v_{\parallel}, v_{\perp 1}, v_{\perp 2}$ 
    
```

2. Regularized Kappa distribution

[Scherer+ 2017]

- Kappa distribution with a high-energy cutoff

$$f_{\text{rk}}(\mathbf{v}; \kappa, \theta, \alpha) d^3v = \frac{N_0}{(\pi \kappa \theta^2)^{3/2} \mathcal{U}\left(\frac{3}{2}, \frac{3}{2} - \kappa, \alpha^2 \kappa\right)} \left(1 + \frac{v^2}{\kappa \theta^2}\right)^{-(\kappa+1)} \exp\left(-\alpha^2 \frac{v^2}{\theta^2}\right) d^3v$$



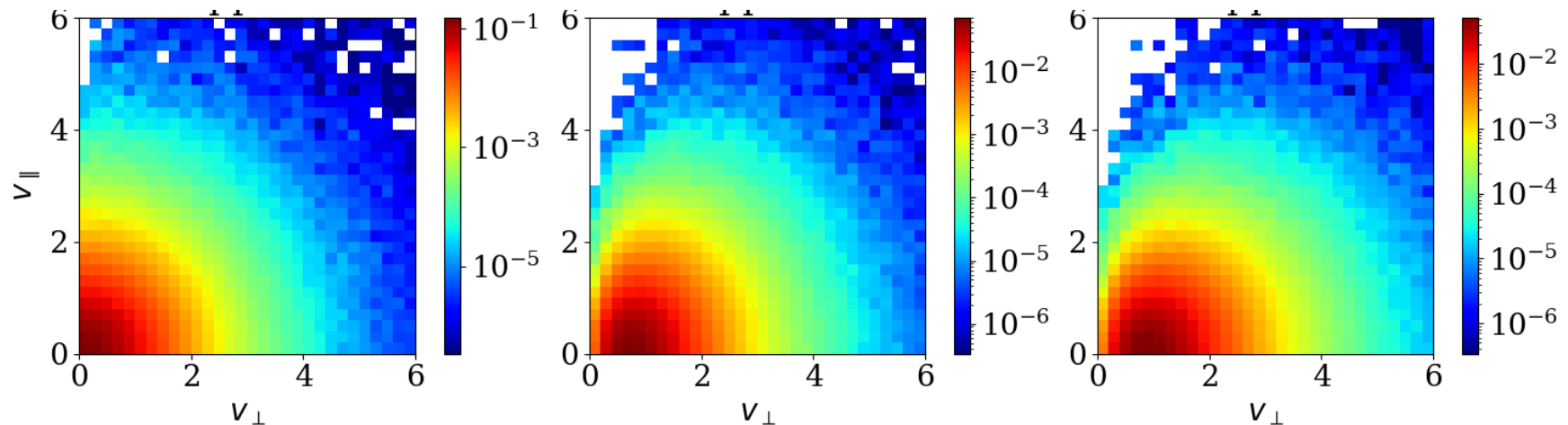
- Two rejection methods are compared

3. Subtracted Kappa distribution

[Summers & Stone 2025]

- Kappa Loss-Cone (KLC) model for a narrow loss-cone

$$f_{\text{sk}}(v_{\parallel}, v_{\perp}) d^3v = N \cdot C_{\text{sk}} \left\{ \frac{1 - \Delta\beta}{1 - \beta} \left(1 + \frac{v_{\parallel}^2}{\kappa\theta_{\parallel}^2} + \frac{v_{\perp}^2}{\kappa\theta_{\perp}^2} \right)^{-(\kappa+1)} - \frac{1 - \Delta}{1 - \beta} \left(1 + \frac{v_{\parallel}^2}{\kappa\theta_{\parallel}^2} + \frac{v_{\perp}^2}{\beta\kappa\theta_{\perp}^2} \right)^{-(\kappa+1)} \right\} d^3v$$



$\beta=0.0$

Standard Kappa distribution

$\beta=0.5$

$\beta=1.0$

Dory/Summers-type
KLC distribution with $j=1$

3. Subtracted Kappa distribution

[Summers & Stone 2025]

- Kappa Loss-Cone (KLC) model for a narrow loss-cone

$$f_{\text{sk}}(v_{\parallel}, v_{\perp}) d^3v = N \cdot C_{\text{sk}} \left\{ \frac{1 - \Delta\beta}{1 - \beta} \left(1 + \frac{v_{\parallel}^2}{\kappa\theta_{\parallel}^2} + \frac{v_{\perp}^2}{\kappa\theta_{\perp}^2} \right)^{-(\kappa+1)} - \frac{1 - \Delta}{1 - \beta} \left(1 + \frac{v_{\parallel}^2}{\kappa\theta_{\parallel}^2} + \frac{v_{\perp}^2}{\beta\kappa\theta_{\perp}^2} \right)^{-(\kappa+1)} \right\} d^3v$$

Algorithm 5: subtracted Kappa

```
generate  $U_1, U_2, U_3 \sim U(0, 1)$     Uniform variate
generate  $N \sim \mathcal{N}(0, 1)$           Normal variate
generate  $Y \sim \text{Ga}(\kappa - 1/2, 2)$     Gamma variate
 $x \leftarrow -\log U_1 - \beta \log \left( \min \left( \frac{U_2}{1-\Delta}, 1 \right) \right)$ 
 $v_{\perp 1} \leftarrow \theta_{\perp} \sqrt{2\kappa x} \cos(2\pi U_3) / \sqrt{Y}$ 
 $v_{\perp 2} \leftarrow \theta_{\perp} \sqrt{2\kappa x} \sin(2\pi U_3) / \sqrt{Y}$ 
 $v_{\parallel} \leftarrow \theta_{\parallel} \sqrt{\kappa} N / \sqrt{Y}$ 
return  $v_{\perp 1}, v_{\perp 2}, v_{\parallel}$ 
```

- Loss-cone filling factor Δ is included
- Formal proof in the arXiv paper

4~7. Ring and shell distributions

[Usami & Horiuchi 2022 FrP, SZ+ arXiv:2510.11890]

• Conventional distributions

- Gaussian width, but recipes are not so simple

$$f_{\text{ring}}(\mathbf{v})d^3v = N_0 \frac{1}{\pi^{3/2} \theta_{\parallel} \theta_{\perp}^2 A_2(\frac{V}{\theta_{\perp}})} \exp\left(-\frac{v_{\parallel}^2}{\theta_{\parallel}^2} - \frac{(v_{\perp} - V)^2}{\theta_{\perp}^2}\right) d^3v,$$

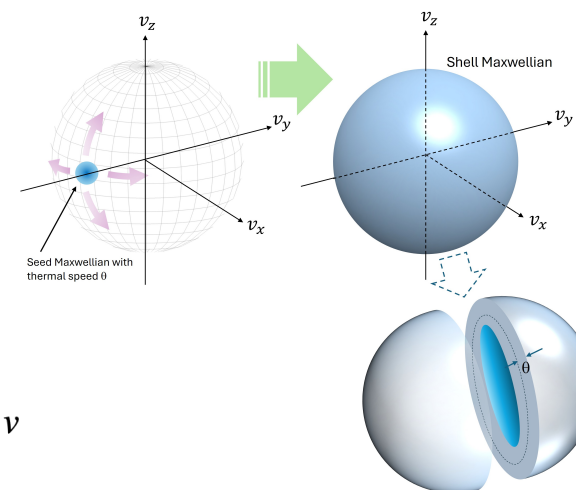
$$f_{\text{shell}}(\mathbf{v})d^3v = \frac{N_0}{2\pi\theta^3 A_3(\frac{V}{\theta})} \exp\left(-\frac{(|\mathbf{v}| - V)^2}{\theta^2}\right) d^3v$$

• Our proposals

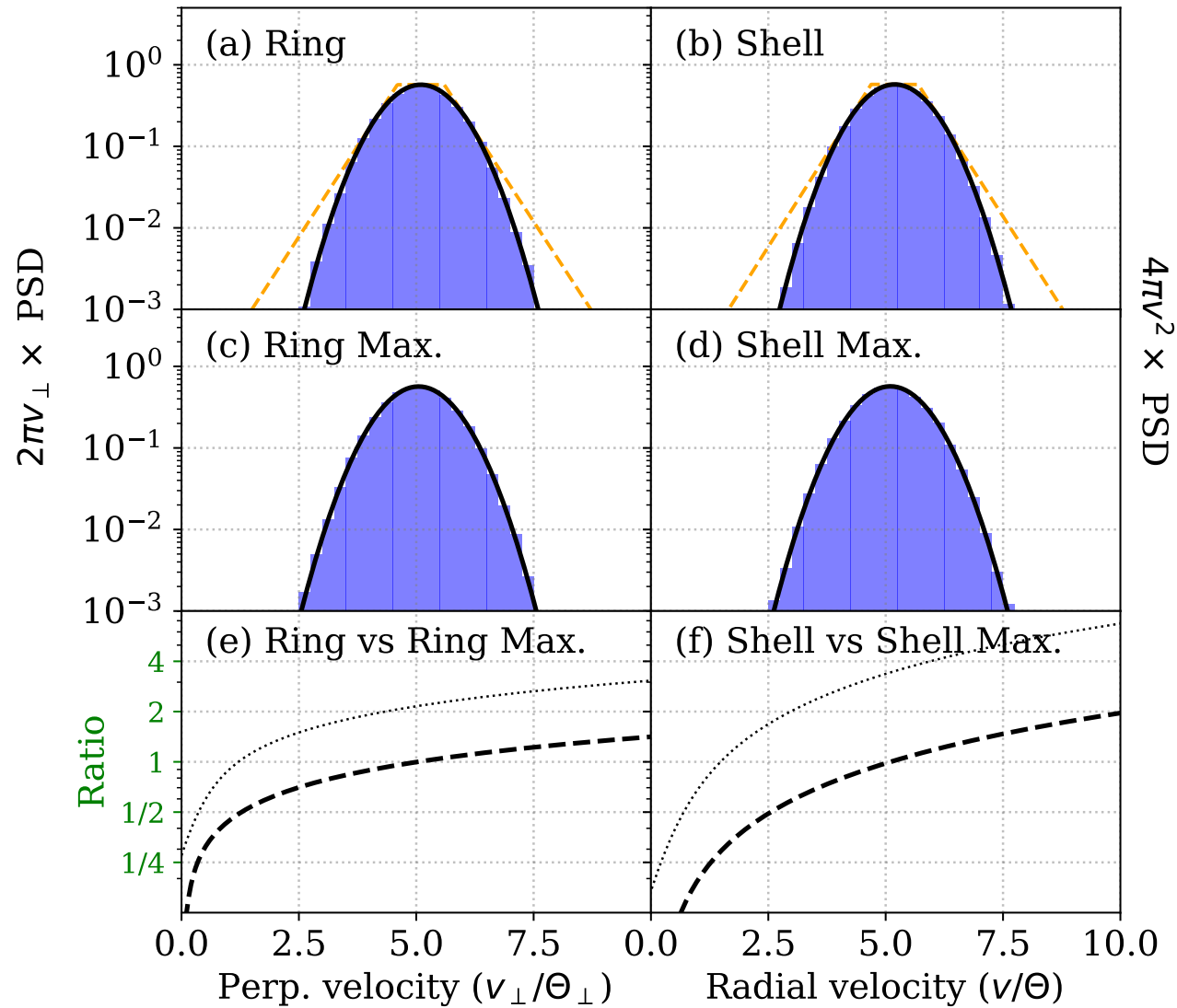
- Based on scattering of a seed population

$$f_{\text{rM}}(\mathbf{v})d^3v = \frac{N_0}{\pi^{3/2} \theta_{\parallel} \theta_{\perp}^2} \exp\left(-\frac{v_{\parallel}^2}{\theta_{\parallel}^2} - \frac{v_{\perp}^2 + V^2}{\theta_{\perp}^2}\right) I_0\left(\frac{2v_{\perp}V}{\theta_{\perp}^2}\right) d^3v$$

$$f_{\text{sM}}(\mathbf{v})d^3v = \frac{N_0}{4rV\theta(\pi)^{3/2}} \left\{ \exp\left(-\frac{(r - V)^2}{\theta^2}\right) - \exp\left(-\frac{(r + V)^2}{\theta^2}\right) \right\} d^3v$$



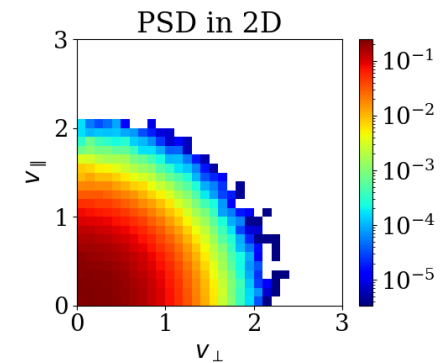
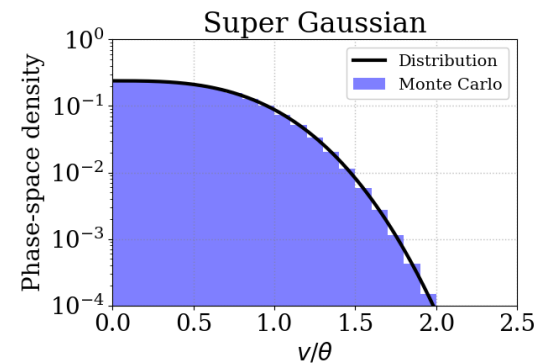
4~7. Ring & shell distributions



8 & 9. Isotropic distributions

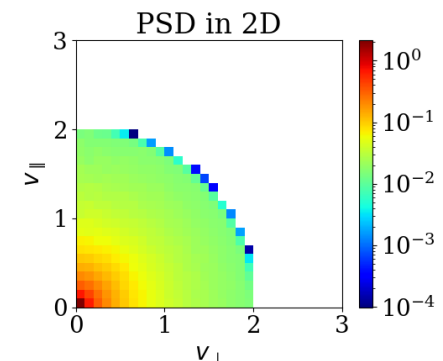
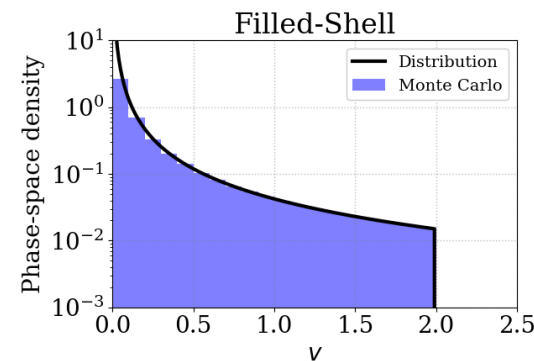
- Super Gaussian (Dum 1974)

$$\frac{Np}{4\pi\theta^3\Gamma(3/p)} \exp\left\{-\left(\frac{v}{\theta}\right)^p\right\}$$



- Filled-Shell distribution (popular in heliophysics)

$$\frac{N(3+p)}{4\pi V^{3+p}} v^p \cdot \mathcal{H}(V-v)$$



- For recipes, please see our arXiv paper

Summary

- Numerical recipes for nine VDFs
 - 1. (r,q) distribution (including flattop)
 - 2. Regularized Kappa distribution
 - 3. Subtracted Kappa distribution
 - 4 & 5. Ring and shell distributions
 - 6 & 7. Ring and shell Maxwellians
 - 8. Super Gaussian distribution
 - 9. Filled-shell distribution
- Reference:
 - S. Zenitani, S. Usami, & S. Matsukiyo, *submitted to JGR*, arxiv:2510.11890
 - <https://arxiv.org/abs/2510.11890>