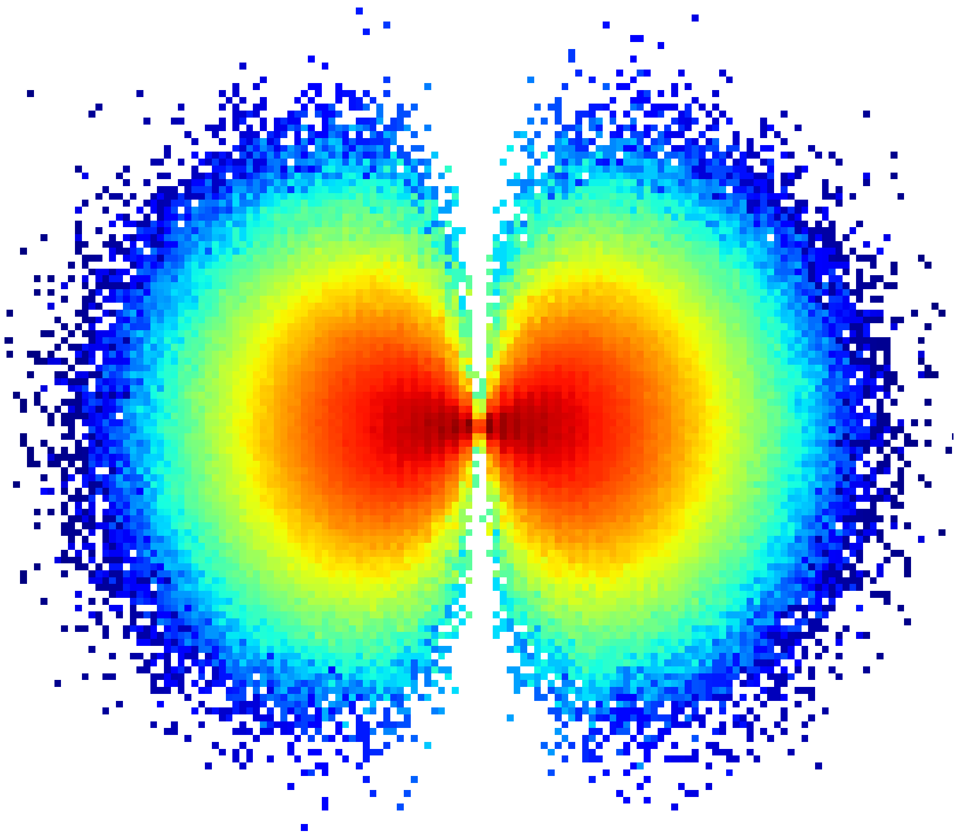
Loading loss-cone distributions in particle simulations



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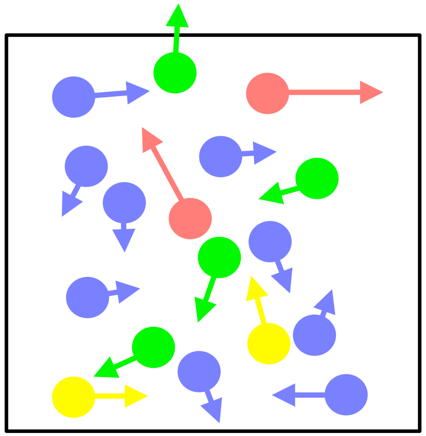
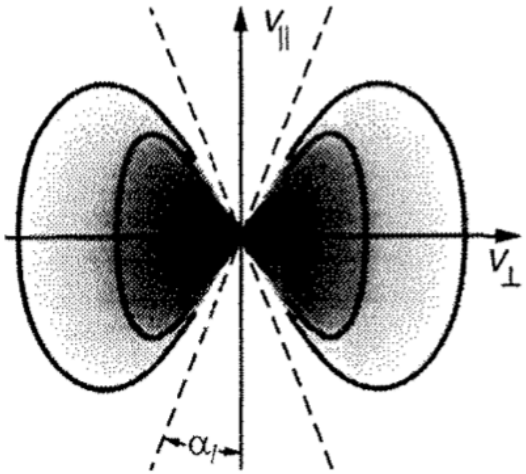
Velocity distribution in particle simulation



- Maxwell distribution
  - Many algorithms are known
  - e.g. Box=Muller (1958) method

$$n_1 = \sqrt{-2\ln U_1}\cos(2\pi U_2)$$
$$n_2 = \sqrt{-2\ln U_1}\sin(2\pi U_2)$$



- What about loss-cone distributions?

We offer numerical recipes for generating loss-cone distributions

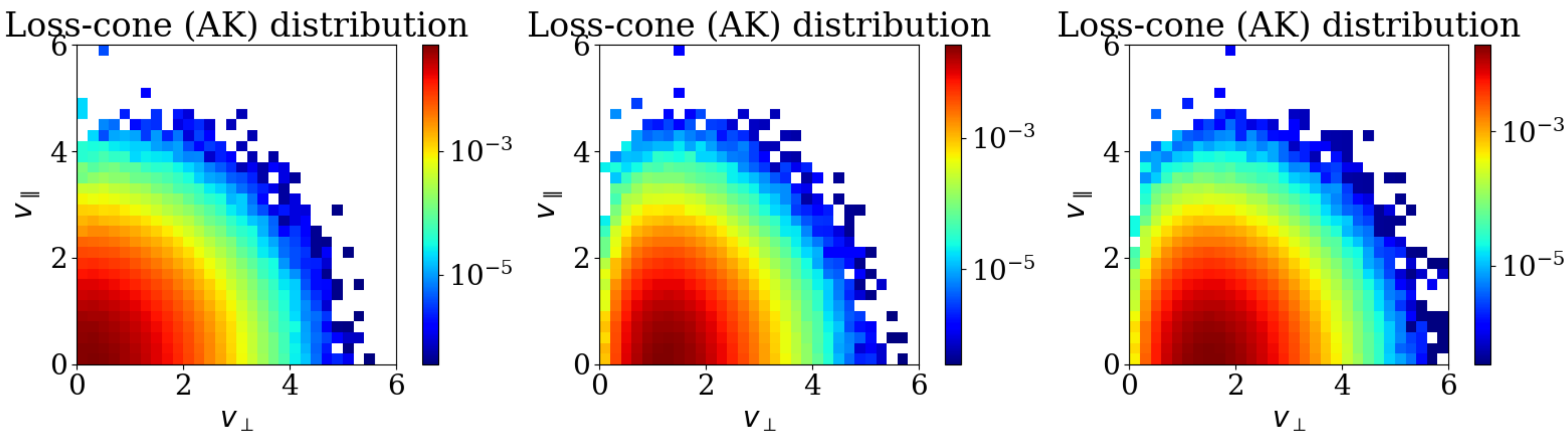
Baumjohann & Treumann, 1996

Subtracted Maxwellian

[Ashour-Abdalla & Kennel, 1978]

$$f(v_\parallel, v_\perp) = \frac{N_0}{\pi^{1/2}\theta_\parallel} \exp\left(-\frac{v_\parallel^2}{\theta_\parallel^2}\right) \times \frac{1}{\pi\theta_\perp^2(1-\beta)} \left\{\exp\left(-\frac{v_\perp^2}{\theta_\perp^2}\right) - \exp\left(-\frac{v_\perp^2}{\beta\theta_\perp^2}\right)\right\}$$

$$P_\parallel = \frac{1}{2} N_0 m \theta_\parallel^2, \quad P_\perp = \frac{1}{2} N_0 m \theta_\perp^2 (1+\beta)$$

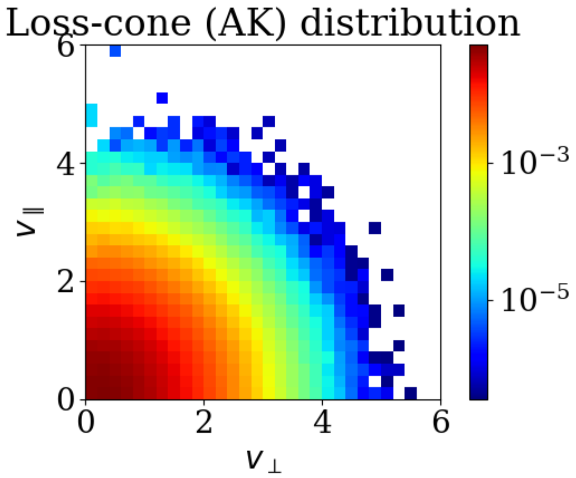


β=0.0 β=0.5 β=1.0

Subtracted Maxwellian

$$f(v_\parallel, v_\perp) = \frac{N_0}{\pi^{1/2}\theta_\parallel} \exp\left(-\frac{v_\parallel^2}{\theta_\parallel^2}\right) \times \frac{1}{\pi\theta_\perp^2(1-\beta)} \left\{\exp\left(-\frac{v_\perp^2}{\theta_\perp^2}\right) - \exp\left(-\frac{v_\perp^2}{\beta\theta_\perp^2}\right)\right\}$$

$$x \equiv v_\perp^2/\theta_\perp^2 \qquad f_X(x) = \frac{1}{1-\beta}\left(\exp(-x) - \exp\left(-\frac{x}{\beta}\right)\right)$$



β=0.0

- Exponential distribution

$$f_X(x) = e^{-x}$$

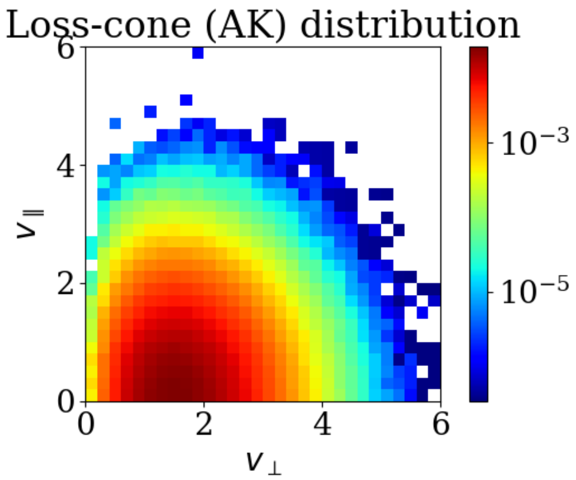
$$x \leftarrow -\log U_1$$

U_1: Uniform random variate

Subtracted Maxwellian

$$f(v_\parallel, v_\perp) = \frac{N_0}{\pi^{1/2}\theta_\parallel} \exp\left(-\frac{v_\parallel^2}{\theta_\parallel^2}\right) \times \frac{1}{\pi\theta_\perp^2(1-\beta)} \left\{\exp\left(-\frac{v_\perp^2}{\theta_\perp^2}\right) - \exp\left(-\frac{v_\perp^2}{\beta\theta_\perp^2}\right)\right\}$$

$$x \equiv v_\perp^2/\theta_\perp^2 \qquad f_X(x) = \frac{1}{1-\beta}\left(\exp(-x) - \exp\left(-\frac{x}{\beta}\right)\right)$$



β=1.0

- Gamma (Erlang) distribution

$$\lim_{\beta\to1}\left(\frac{e^{-x} - e^{-x/\beta}}{1-\beta}\right) = \left(\frac{d}{d\beta}e^{-x/\beta}\right)\Bigg|_{\beta=1} = xe^{-x}$$

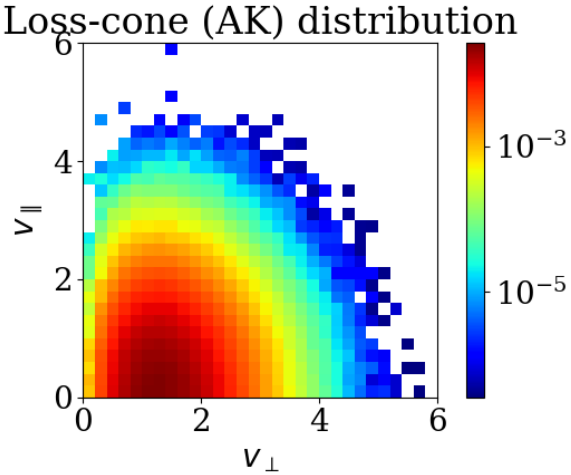
$$x \leftarrow -\log U_1 - \log U_2$$

U_1 , U_2: Uniform random variates

Subtracted Maxwellian

$$f(v_\parallel, v_\perp) = \frac{N_0}{\pi^{1/2}\theta_\parallel} \exp\left(-\frac{v_\parallel^2}{\theta_\parallel^2}\right) \times \frac{1}{\pi\theta_\perp^2(1-\beta)} \left\{\exp\left(-\frac{v_\perp^2}{\theta_\perp^2}\right) - \exp\left(-\frac{v_\perp^2}{\beta\theta_\perp^2}\right)\right\}$$

$$x \equiv v_\perp^2/\theta_\perp^2 \qquad f_X(x) = \frac{1}{1-\beta}\left(\exp(-x) - \exp\left(-\frac{x}{\beta}\right)\right)$$



$0.0<\beta<1.0$

- Consistent with two limits

$$x \leftarrow -\log U_1 - \beta \log U_2$$

$$s \sim G_s(s) = \exp(-s), \quad s \geq 0$$

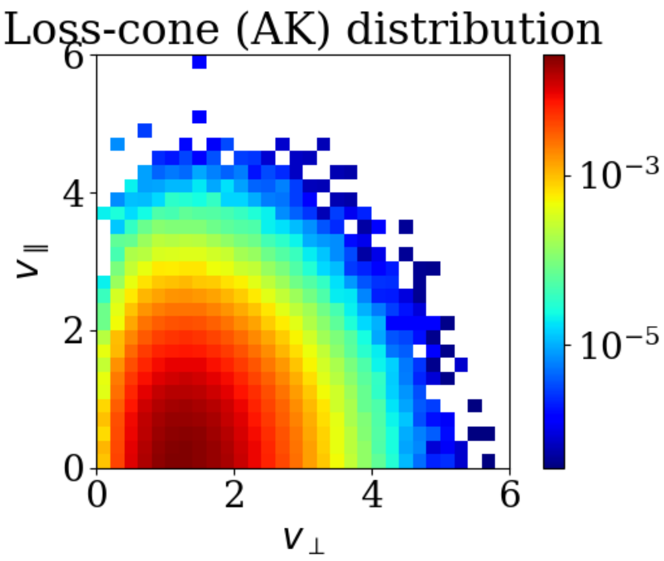
$$t \sim G_t(t) = \frac{1}{\beta}\exp\left(-\frac{t}{\beta}\right), \quad t \geq 0$$

$$G(x) = \int_0^x G_s(s)\cdot G_t(x-s)\, ds = \frac{1}{\beta}\int_0^x \exp\left(-\frac{\beta-1}{\beta}s - \frac{x}{\beta}\right) ds$$

$$= \frac{1}{1-\beta}\left\{\exp(-x) - \exp\left(-\frac{x}{\beta}\right)\right\}$$

Subtracted Maxwellian - Recipe

$$f(v_\parallel, v_\perp) = \frac{N_0}{\pi^{1/2}\theta_\parallel} \exp\left(-\frac{v_\parallel^2}{\theta_\parallel^2}\right) \times \frac{1}{\pi\theta_\perp^2(1-\beta)} \left\{\exp\left(-\frac{v_\perp^2}{\theta_\perp^2}\right) - \exp\left(-\frac{v_\perp^2}{\beta\theta_\perp^2}\right)\right\}$$



Algorithm 2

generate $U_1, U_2, U_3 \sim U(0,1)$ Uniform variate

generate $N \sim \mathcal{N}(0,1)$ Normal variate

$x \leftarrow -\log U_1 - \beta \log U_2$

$v_{\perp 1} \leftarrow \theta_\perp \sqrt{x} \cos(2\pi U_3)$

$v_{\perp 2} \leftarrow \theta_\perp \sqrt{x} \sin(2\pi U_3)$

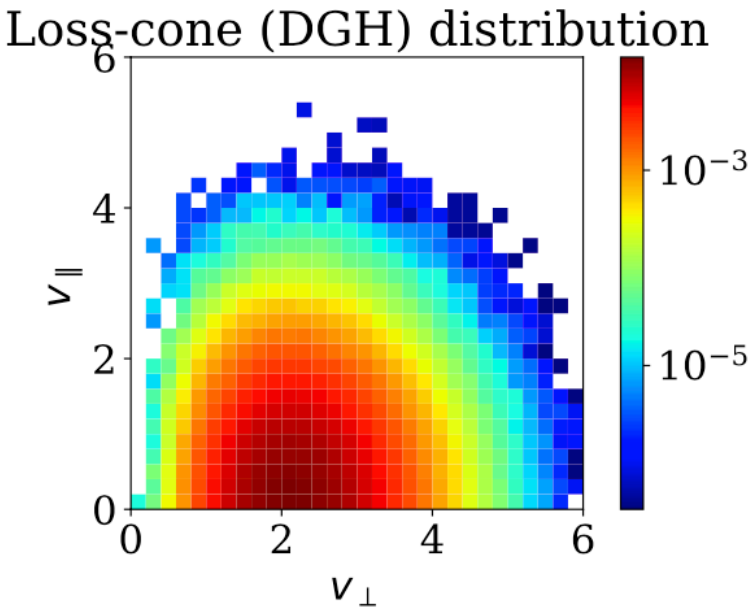
$v_\parallel \leftarrow \theta_\parallel \sqrt{1/2}\, N$

return $v_{\perp 1}, v_{\perp 2}, v_\parallel$

Dory-type loss-cone distribution

[Dory et al., 1965]

$$f(v_\parallel, v_\perp) = \frac{N_0}{\pi^{3/2}\theta_\parallel \theta_\perp^2 \Gamma(j+1)} \left(\frac{v_\perp}{\theta_\perp}\right)^{2j} \exp\left(-\frac{v_\parallel^2}{\theta_\parallel^2} - \frac{v_\perp^2}{\theta_\perp^2}\right)$$



Algorithm 3

generate $X \sim \mathrm{Ga}(j+1, 1)$

// generate uniform $Y_1 \cdots Y_{j+1} \sim U(0,1)$

// $X \leftarrow -\log\left(\prod_{k=1}^{j+1} Y_k\right)$

generate $U \sim U(0,1)$

generate $N \sim \mathcal{N}(0,1)$

$v_{\perp 1} \leftarrow \theta_\perp \sqrt{X} \cos(2\pi U)$

$v_{\perp 2} \leftarrow \theta_\perp \sqrt{X} \sin(2\pi U)$

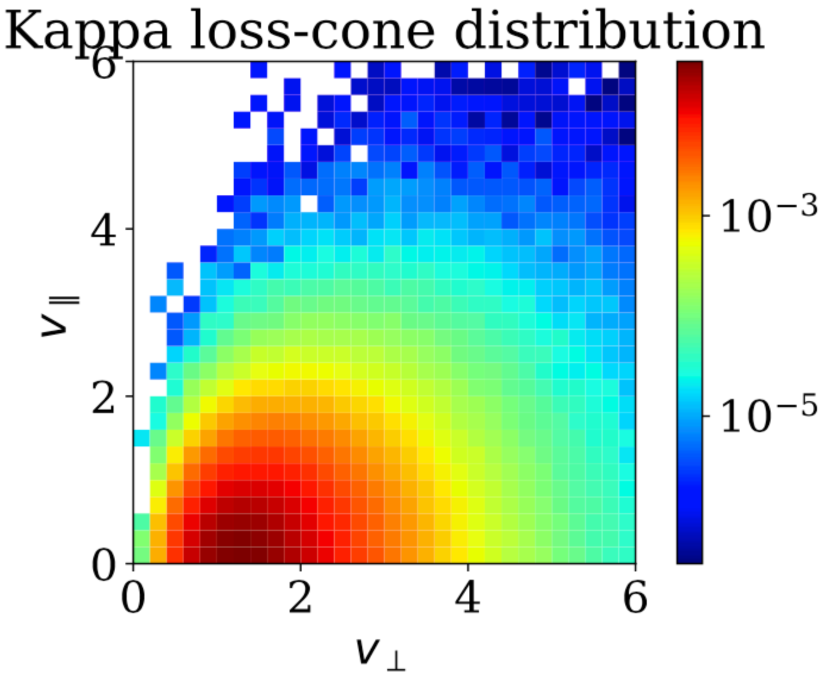
$v_\parallel \leftarrow \theta_\parallel \sqrt{1/2}\, N$

return $v_{\perp 1}, v_{\perp 2}, v_\parallel$

Kappa loss-cone distribution

[Summers & Thorne 1991]

$$f(\boldsymbol{v}) = \frac{N_0}{\pi^{3/2}\theta_\parallel \theta_\perp^2 \kappa^{j+3/2}} \frac{\Gamma(\kappa+j+1)}{\Gamma(j+1)\Gamma(\kappa-1/2)} \times \left(\frac{v_\perp}{\theta_\perp}\right)^{2j} \left(1 + \frac{{v_\parallel}^2}{\kappa\theta_\parallel^2} + \frac{{v_\perp}^2}{\kappa\theta_\perp^2}\right)^{-(\kappa+j+1)}$$



Algorithm 4

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generate $N \sim \mathcal{N}(0, 1)$

generate $Y \sim \mathrm{Ga}(\kappa - 1/2, 2)$ Gamma variate

generate $X \sim \mathrm{Ga}(j + 1, 2)$ Gamma variate

generate $U \sim U(0, 1)$

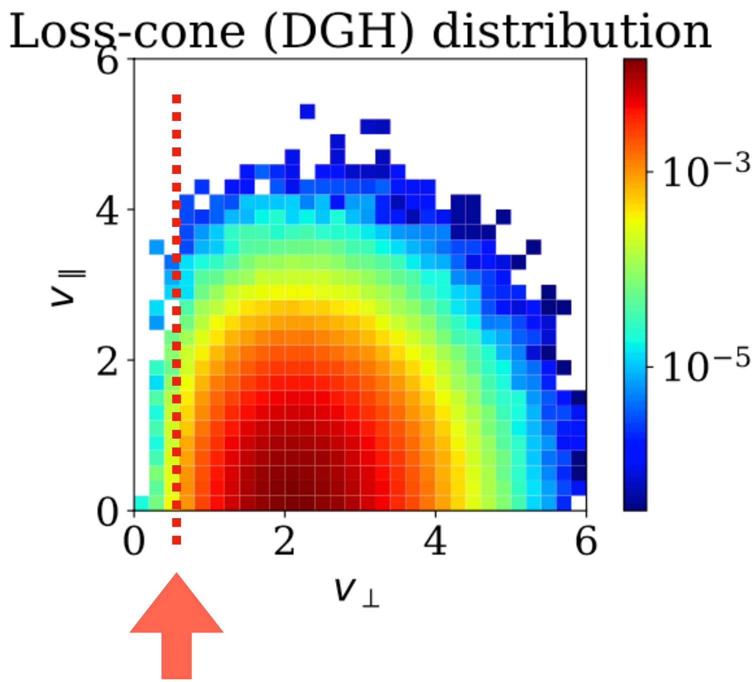
$v_{\perp 1} \leftarrow \theta_\perp \sqrt{\kappa X} \cos(2\pi U)/\sqrt{Y}$

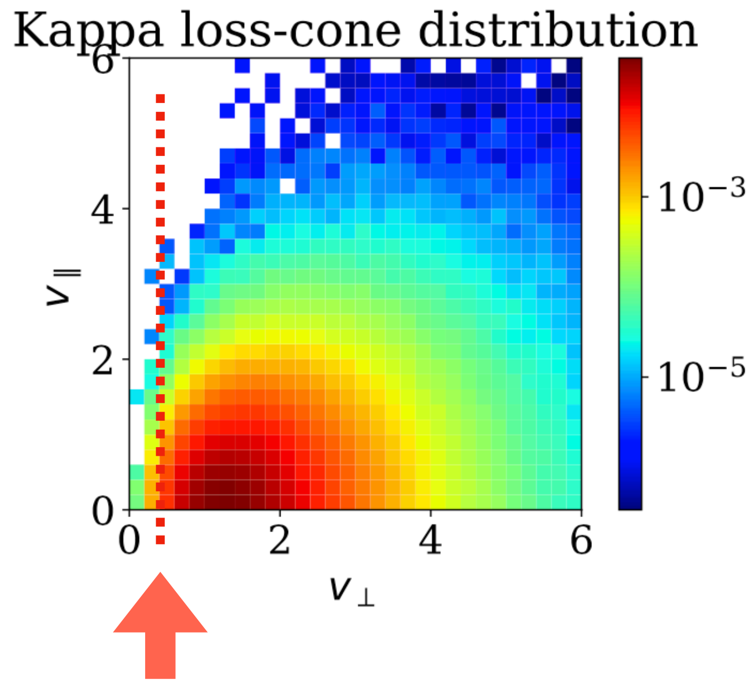
$v_{\perp 2} \leftarrow \theta_\perp \sqrt{\kappa X} \sin(2\pi U)/\sqrt{Y}$

$v_\parallel \leftarrow \theta_\parallel \sqrt{\kappa} N/\sqrt{Y}$

return $v_{\perp 1}, v_{\perp 2}, v_\parallel$

Loss-cone?

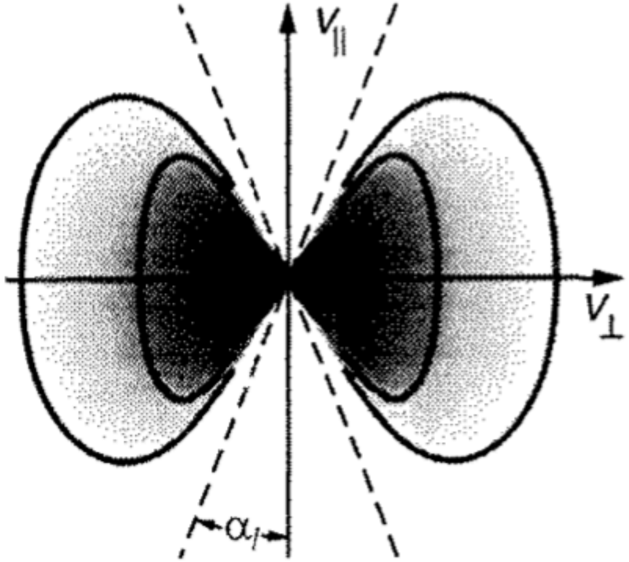




$$f(v_\parallel, v_\perp) = \frac{N_0}{\pi^{3/2}\theta_\parallel \theta_\perp^2 \Gamma(j+1)} \left(\frac{v_\perp}{\theta_\perp}\right)^{2j} \exp\left(-\frac{v_\parallel^2}{\theta_\parallel^2} - \frac{v_\perp^2}{\theta_\perp^2}\right)$$

$$f(\boldsymbol{v}) = \frac{N_0}{\pi^{3/2}\theta_\parallel \theta_\perp^2 \kappa^{j+3/2}} \frac{\Gamma(\kappa+j+1)}{\Gamma(j+1)\Gamma(\kappa-1/2)} \times \left(\frac{v_\perp}{\theta_\perp}\right)^{2j} \left(1 + \frac{{v_\parallel}^2}{\kappa\theta_\parallel^2} + \frac{{v_\perp}^2}{\kappa\theta_\perp^2}\right)^{-(\kappa+j+1)}$$

Pitch-angle (PA) type distribution

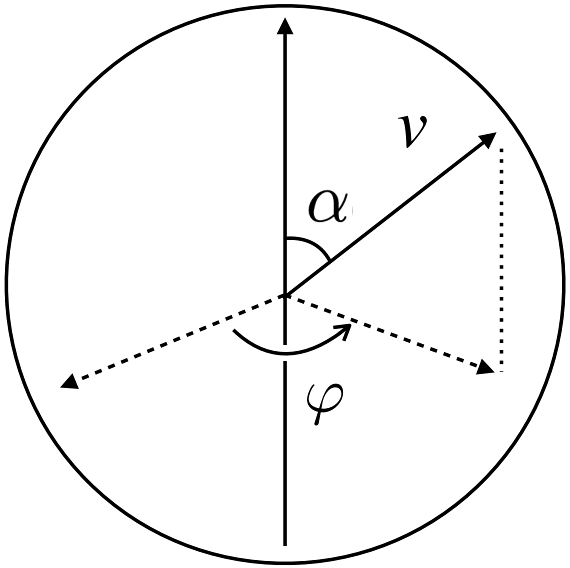


$$\propto \Big(\sin\alpha \Big)^{2j}$$

$$\Leftrightarrow \quad \propto \left(\frac{v_\perp}{v}\right)^{2j}$$

c.f. Kennel 1966

Utilizing Beta distribution



$$\iiint f_0(v)\ (\sin\alpha)^{2j}\, d^3v$$

$$= 4\pi\left(\int_0^\infty v^2 f_0(v)\ dv\right)\left(\int_0^{\pi/2} (\sin\alpha)^{2j+1}\, d\alpha\right)$$

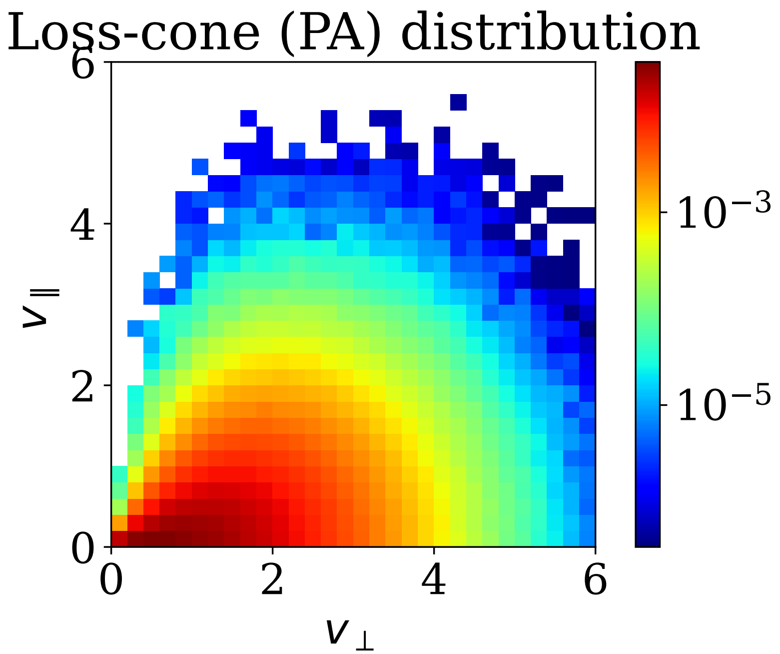
$$x \equiv \cos^2\alpha$$

$$\longrightarrow \frac{B(1/2, j+1)}{2}\left\{\int_0^1 \frac{(1-x)^j x^{-1/2}}{B(1/2, j+1)} dx\right\}$$

Beta distribution

- One can transform isotropic distributions to loss-cone distributions via Beta random variate.

PA-type loss-cone distribution



Algorithm 5.3: Loss-cone distribution

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generate $N \sim \mathcal{N}(0, 1)$

generate $X_1 \sim \mathrm{Ga}(3/2, 1)$

generate $X_2 \sim \mathrm{Ga}(j + 1, 2)$

generate $U \sim U(0, 1)$

$v_{\perp 1} \leftarrow \theta \sqrt{X_1} \sqrt{\frac{X_2}{N^2 + X_2}} \cos(2\pi U)$

$v_{\perp 2} \leftarrow \theta \sqrt{X_1} \sqrt{\frac{X_2}{N^2 + X_2}} \sin(2\pi U)$

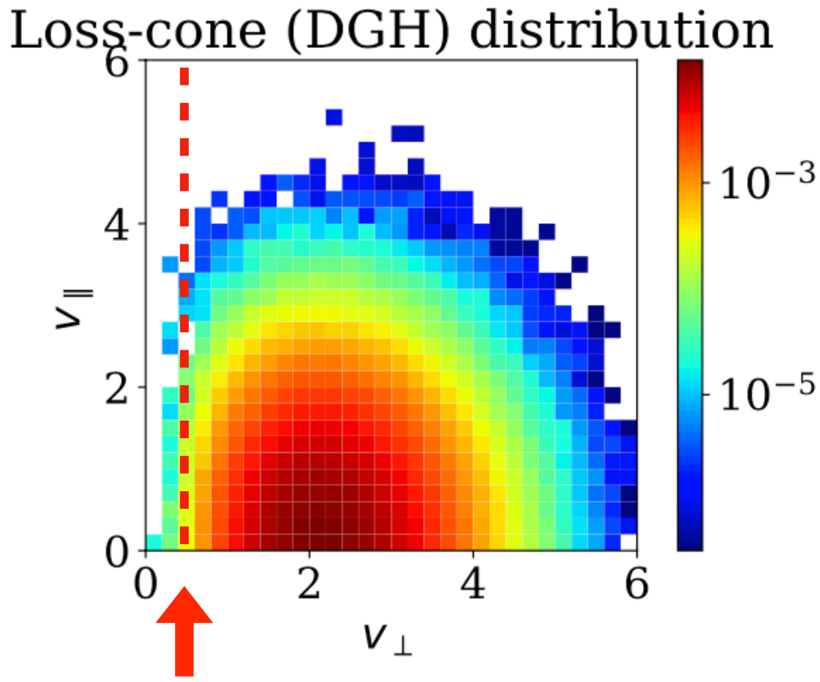
$v_\parallel \leftarrow \theta \sqrt{X_1} \frac{N}{\sqrt{N^2 + X_2}}$

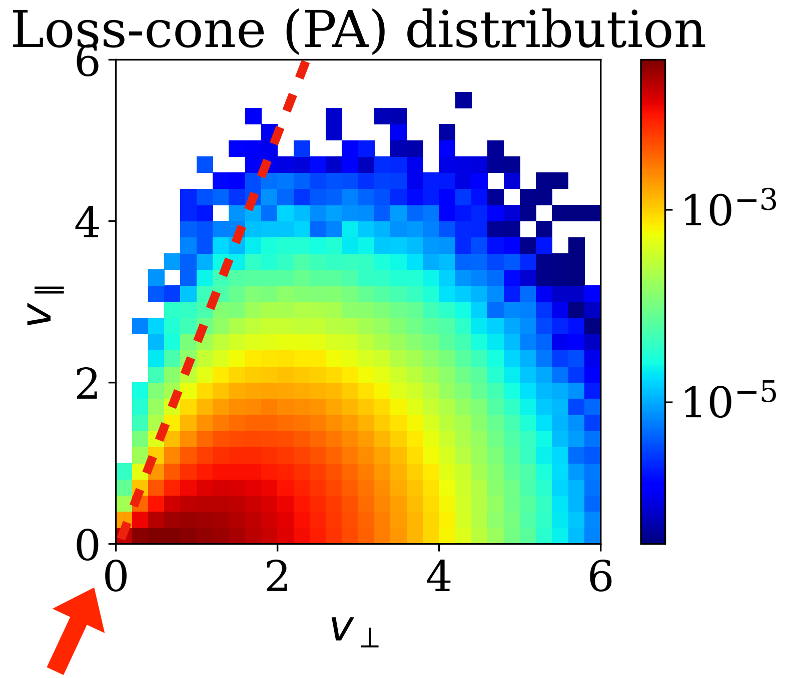
return $v_{\perp 1}, v_{\perp 2}, v_\parallel$

Loss-cone transform (Beta variate)

$$f(\boldsymbol{v}) = \frac{N_0}{\pi^2 \theta^3} \frac{2\Gamma(j + 3/2)}{\Gamma(j + 1)} \left(\frac{v_\perp}{v} \right)^{2j} \exp\left(- \frac{v^2}{\theta^2} \right)$$

Old ($V_\perp$-type) vs New (PA-type)





$$f(v_\parallel, v_\perp) = \frac{N_0}{\pi^{3/2}\theta_\parallel \theta_\perp^2 \Gamma(j+1)} \left(\frac{v_\perp}{\theta_\perp}\right)^{2j} \exp\left(-\frac{v_\parallel^2}{\theta_\parallel^2} - \frac{v_\perp^2}{\theta_\perp^2}\right)$$

$$f(\boldsymbol{v}) = \frac{N_0}{\pi^2\theta^3} \frac{2\Gamma(j+3/2)}{\Gamma(j+1)} \left(\frac{v_\perp}{v}\right)^{2j} \exp\left(-\frac{v^2}{\theta^2}\right)$$

Summary

- 1. Numerical procedures for loss-cone distributions
  - Subtracted Maxwellian
  - Dory-type loss-cone distribution
  - Kappa loss-cone distribution
- 2. Pitch-angle type distributions
  - Loss-cone transform, via Beta random variate
  - PA-type loss-cone distributions
- Advantages (not presented)
  - Flexibility - Loss-cone index j and kappa index κ can be non-integer
  - SIMD/SIMT friendly - No branching, no rejection
- Reference:

  - S. Zenitani & S. Nakano, Loading Loss-Cone Distributions in Particle Simulations, JGR: Space Physics 128, e2023JA031983 (arXiv:2309.06879)