

Hyper Boris integrators for particle-in-cell simulation

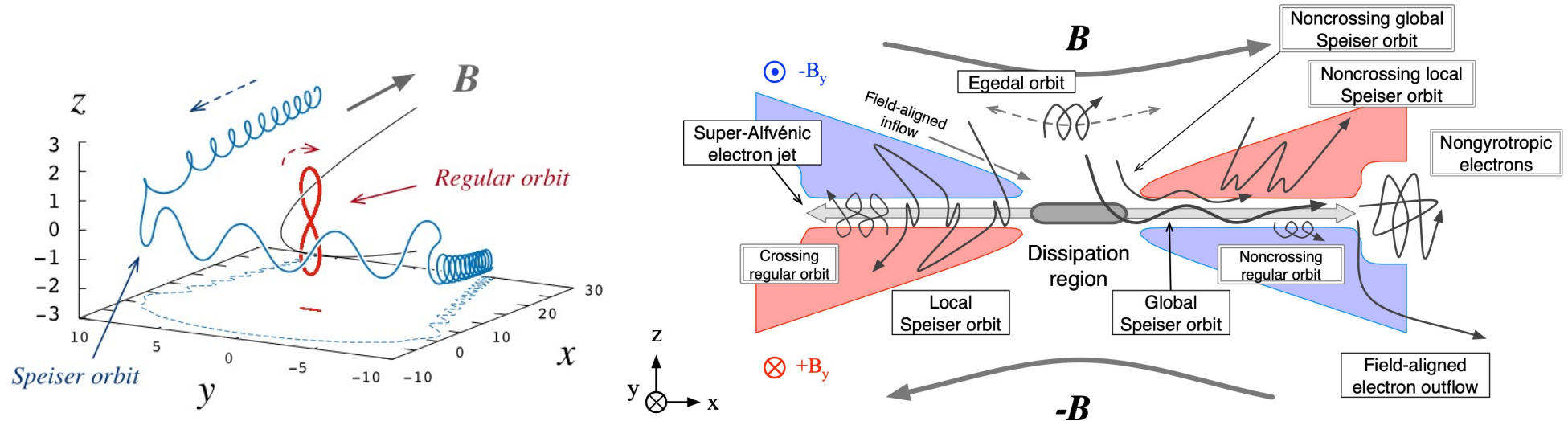
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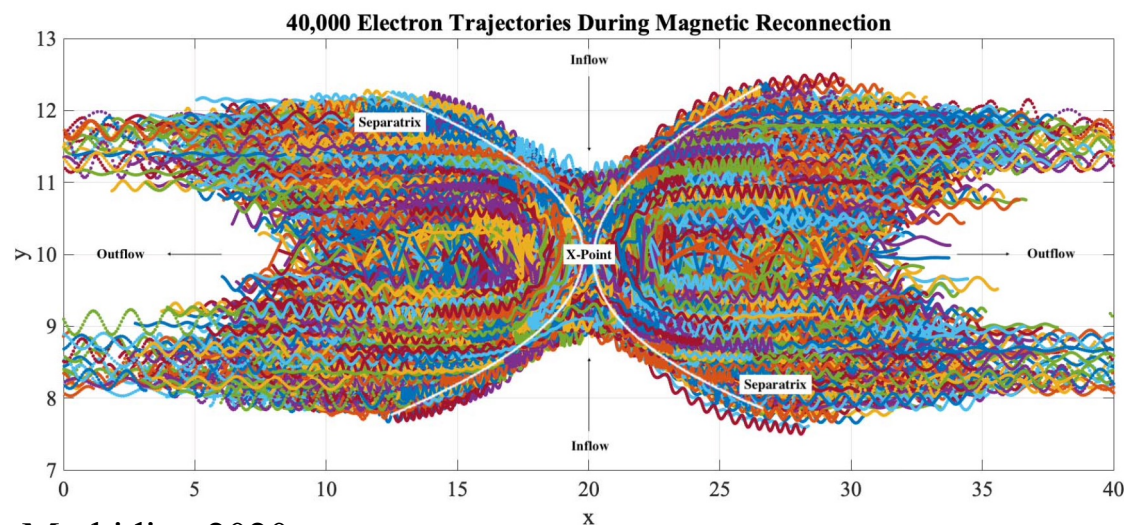
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Particle motion in reconnection



Zenitani & Nagai 2016



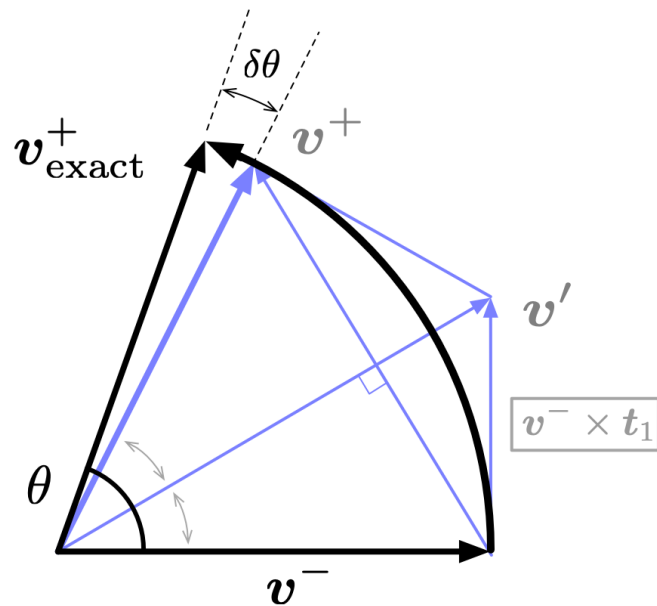
It is important to accurately solve particle motion in PIC simulation

Markidis+ 2020

Boris solver (a.k.a. Buneman-Boris solver)

$$\frac{\mathbf{x}^{t+\Delta t} - \mathbf{x}^t}{\Delta t} = \mathbf{v}^{t+\frac{1}{2}\Delta t}$$

$$m \frac{\mathbf{v}^{t+\frac{1}{2}\Delta t} - \mathbf{v}^{t-\frac{1}{2}\Delta t}}{\Delta t} = q \left(\mathbf{E}^t + \frac{\mathbf{v}^{t+\frac{1}{2}\Delta t} + \mathbf{v}^{t-\frac{1}{2}\Delta t}}{2} \times \mathbf{B}^t \right)$$



element E vector

element B vector

$$\mathbf{e}_1 \equiv \frac{q\Delta t}{2m} \mathbf{E}, \quad \mathbf{t}_1 \equiv \frac{q\Delta t}{2m} \mathbf{B}$$

$$\mathbf{v}^- = \mathbf{v}^{t-\frac{1}{2}\Delta t} + \mathbf{e}_1$$

Half acc. by E

$$\mathbf{v}' = \mathbf{v}^- + \mathbf{v}^- \times \mathbf{t}_1$$

Approximate gyration

$$\mathbf{v}^+ = \mathbf{v}^- + \frac{2}{1 + t_1^2} \mathbf{v}' \times \mathbf{t}_1$$

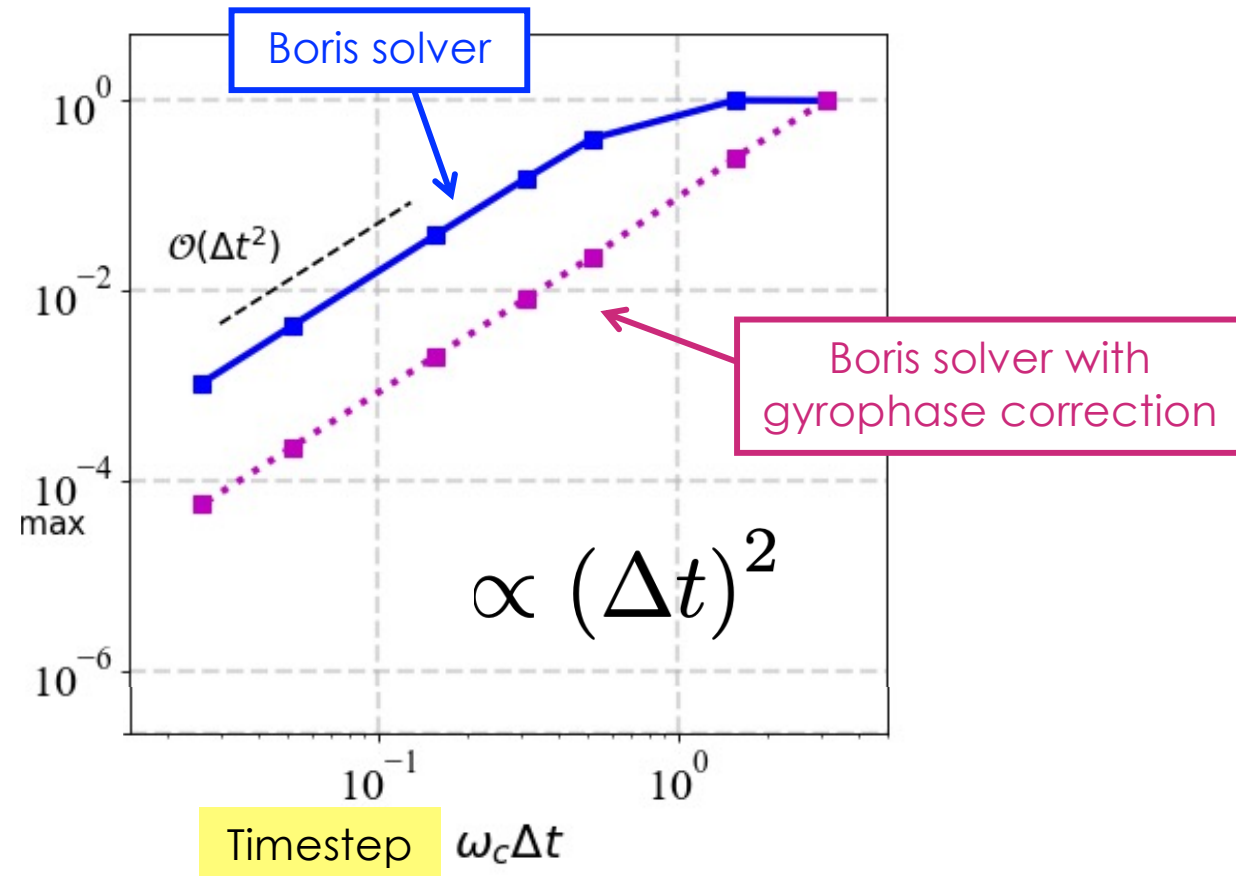
$$\mathbf{v}^{t+\frac{1}{2}\Delta t} = \mathbf{v}^+ + \mathbf{e}_1$$

Half acc. by E

Boris solver: 2nd-order accuracy

Numerical error

$$\|\mathbf{v}_{\text{sim}} - \mathbf{v}_{\text{exact}}\|$$



We propose 3 solutions
to improve the Boris solver

Solution 1: Subcycling

- We repeat the procedure n times with $\frac{\Delta t}{n}$
- E & B are fixed

element E vector element B vector

$$\mathbf{e}_1 \equiv \frac{q\Delta t}{2m} \mathbf{E}, \quad \mathbf{t}_1 \equiv \frac{q\Delta t}{2m} \mathbf{B}$$

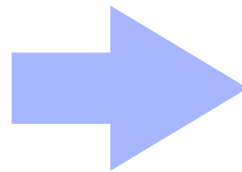
$$\mathbf{v}^- = \mathbf{v}^{t-\frac{1}{2}\Delta t} + \mathbf{e}_1$$

$$\mathbf{v}' = \mathbf{v}^- + \mathbf{v}^- \times \mathbf{t}_1$$

$$\mathbf{v}^+ = \mathbf{v}^- + \frac{2}{1+t_1^2} \mathbf{v}' \times \mathbf{t}_1$$

$$\mathbf{v}^{t+\frac{1}{2}\Delta t} = \mathbf{v}^+ + \mathbf{e}_1$$

subcycling



element E vector element B vector

$$\mathbf{e}_n \equiv \frac{q\Delta t}{2nm} \mathbf{E}, \quad \mathbf{t}_n \equiv \frac{q\Delta t}{2nm} \mathbf{B}$$

$$\left\{ \begin{array}{l} \mathbf{v}^- = \mathbf{v}^{(0)} + \mathbf{e}_n \\ \mathbf{v}' = \mathbf{v}^- + \mathbf{v}^- \times \mathbf{t}_n \\ \mathbf{v}^+ = \mathbf{v}^- + \frac{2}{1+t_n^2} \mathbf{v}' \times \mathbf{t}_n \\ \mathbf{v}^{(1)} = \mathbf{v}^+ + \mathbf{e}_n \\ \vdots \end{array} \right\} \times n \text{ times}$$

$$\left\{ \begin{array}{l} \mathbf{v}^- = \mathbf{v}^{(n-1)} + \mathbf{e}_n \\ \mathbf{v}' = \mathbf{v}^- + \mathbf{v}^- \times \mathbf{t}_n \\ \mathbf{v}^+ = \mathbf{v}^- + \frac{2}{1+t_n^2} \mathbf{v}' \times \mathbf{t}_n \\ \mathbf{v}^{(n)} = \mathbf{v}^+ + \mathbf{e}_n \end{array} \right\}$$

Multicycle formula for arbitrary n

1-cycle

$$\begin{aligned} \mathbf{v}^{t+\frac{1}{2}\Delta t} = & \frac{1-t_1^2}{1+t_1^2} \mathbf{v}^{t-\frac{1}{2}\Delta t} + \frac{2}{1+t_1^2} (\mathbf{v}^{t-\frac{1}{2}\Delta t} \times \mathbf{t}_1) + \frac{2}{1+t_1^2} (\mathbf{v}^{t-\frac{1}{2}\Delta t} \cdot \mathbf{t}_1) \mathbf{t}_1 \\ & + \frac{2}{1+t_1^2} \mathbf{e}_1 + \frac{2}{1+t_1^2} (\mathbf{e}_1 \times \mathbf{t}_1) + \frac{2}{1+t_1^2} (\mathbf{e}_n \cdot \mathbf{t}_1) \mathbf{t}_1 \end{aligned}$$

2-cycles

$$\begin{aligned} \mathbf{v}^{t+\frac{1}{2}\Delta t} = & \frac{1-6t_2^2+t_2^4}{(1+t_2^2)^2} \mathbf{v}^{t-\frac{1}{2}\Delta t} + \frac{4(1-t_2^2)}{(1+t_2^2)^2} (\mathbf{v}^{t-\frac{1}{2}\Delta t} \times \mathbf{t}_2) + \frac{8}{(1+t_2^2)^2} (\mathbf{v}^{t-\frac{1}{2}\Delta t} \cdot \mathbf{t}_2) \mathbf{t}_2 \\ & + \frac{4(1-t_2^2)}{(1+t_2^2)^2} \mathbf{e}_2 + \frac{8}{(1+t_2^2)^2} (\mathbf{e}_2 \times \mathbf{t}_2) + \frac{4(3+t_2^2)}{(1+t_2^2)^2} (\mathbf{e}_2 \cdot \mathbf{t}_2) \mathbf{t}_2 \end{aligned}$$

n-cycles

$$\begin{aligned} \mathbf{v}^{t+\Delta t} = & c_{n1} \mathbf{v}^t + c_{n2} (\mathbf{v}^t \times \mathbf{t}_n) + c_{n3} (\mathbf{v}^t \cdot \mathbf{t}_n) \mathbf{t}_n \\ & + c_{n4} \mathbf{e}_n + c_{n5} (\mathbf{e}_n \times \mathbf{t}_n) + c_{n6} (\mathbf{e}_n \cdot \mathbf{t}_n) \mathbf{t}_n \end{aligned}$$

element E & B vectors

$$\begin{aligned} \mathbf{e}_n & \equiv \frac{q\Delta t}{2nm} \mathbf{E} \\ \mathbf{t}_n & \equiv \frac{q\Delta t}{2nm} \mathbf{B} \end{aligned}$$

$$c_{n1} = T_n \left(\frac{1-t_n^2}{1+t_n^2} \right)$$

$$c_{n2} = c_{n4} = \frac{2}{1+t_n^2} U_{n-1} \left(\frac{1-t_n^2}{1+t_n^2} \right)$$

$$c_{n6} = \frac{2}{t_n^2} \left(n - \frac{1}{1+t_n^2} U_{n-1} \left(\frac{1-t_n^2}{1+t_n^2} \right) \right)$$

$$c_{n3} = c_{n5}$$

$$= \begin{cases} \frac{2}{1+t_n^2} & (\text{for } n = 1) \\ \frac{2}{1+t_n^2} \left(U_k \left(\frac{1-t_n^2}{1+t_n^2} \right) + U_{k-1} \left(\frac{1-t_n^2}{1+t_n^2} \right) \right)^2 & (\text{for } n = 2k + 1) \\ \frac{8}{(1+t_n^2)^2} \left(U_{k-1} \left(\frac{1-t_n^2}{1+t_n^2} \right) \right)^2 & (\text{for } n = 2k) \end{cases}$$

T_n, U_n : Chebyshev polynomials

Zenitani & Kato 2020, 2025

Reminder: Chebyshev polynomials

First kind [\[edit \]](#)

The first few Chebyshev polynomials of the first kind are [OEIS: A028297](#)

$$T_0(x) = 1$$

$$T_1(x) = x$$

$$T_2(x) = 2x^2 - 1$$

$$T_3(x) = 4x^3 - 3x$$

$$T_4(x) = 8x^4 - 8x^2 + 1$$

$$T_5(x) = 16x^5 - 20x^3 + 5x$$

$$T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1$$

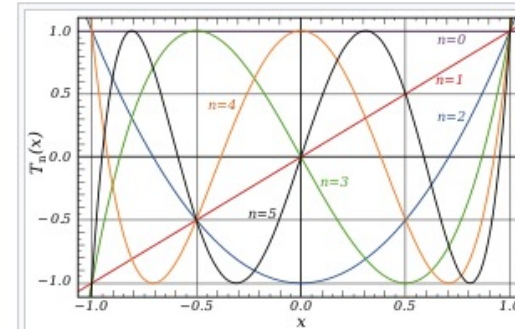
$$T_7(x) = 64x^7 - 112x^5 + 56x^3 - 7x$$

$$T_8(x) = 128x^8 - 256x^6 + 160x^4 - 32x^2 + 1$$

$$T_9(x) = 256x^9 - 576x^7 + 432x^5 - 120x^3 + 9x$$

$$T_{10}(x) = 512x^{10} - 1280x^8 + 1120x^6 - 400x^4 + 50x^2 - 1$$

$$T_{11}(x) = 1024x^{11} - 2816x^9 + 2816x^7 - 1232x^5 + 220x^3 - 11x$$



The first few Chebyshev polynomials of the first kind in the domain $-1 < x < 1$: The flat T_0 , T_1 , T_2 , T_3 , T_4 and T_5 .

Second kind [\[edit \]](#)

The first few Chebyshev polynomials of the second kind are [OEIS: A053117](#)

$$U_0(x) = 1$$

$$U_1(x) = 2x$$

$$U_2(x) = 4x^2 - 1$$

$$U_3(x) = 8x^3 - 4x$$

$$U_4(x) = 16x^4 - 12x^2 + 1$$

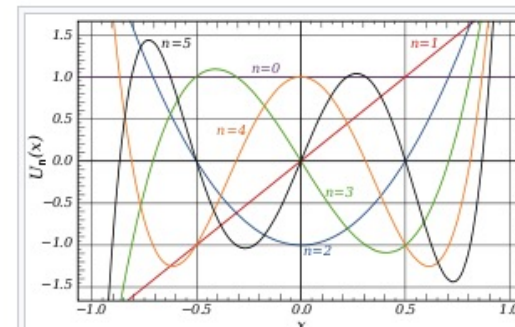
$$U_5(x) = 32x^5 - 32x^3 + 6x$$

$$U_6(x) = 64x^6 - 80x^4 + 24x^2 - 1$$

$$U_7(x) = 128x^7 - 192x^5 + 80x^3 - 8x$$

$$U_8(x) = 256x^8 - 448x^6 + 240x^4 - 40x^2 + 1$$

$$U_9(x) = 512x^9 - 1024x^7 + 672x^5 - 160x^3 + 10x$$

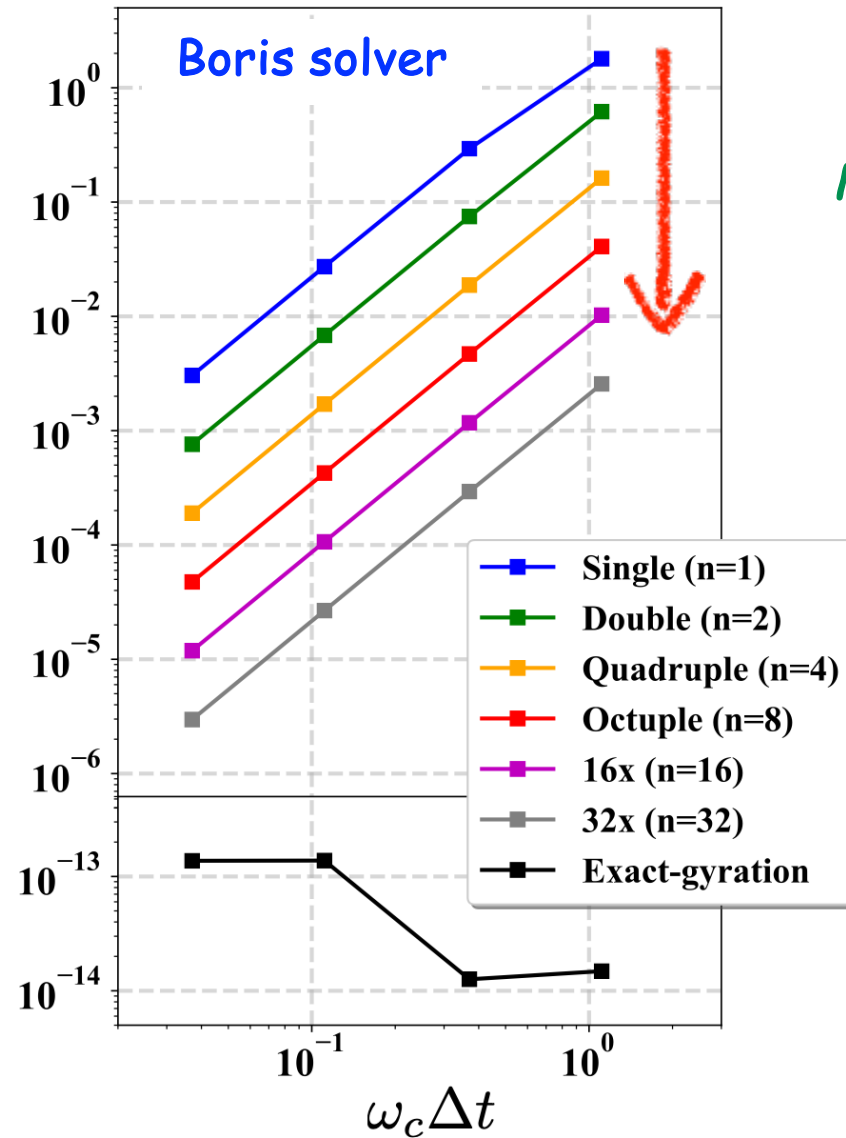


The first few Chebyshev polynomials of the second kind in the domain $-1 < x < 1$: The flat U_0 , U_1 , U_2 , U_3 , U_4 and U_5 . Although not visible in the image, $U_n(1) = n + 1$ and $U_n(-1) = (n + 1)(-1)^n$.

Numerical test

Numerical
error

$$\frac{\|\delta v\|}{\|v\|}$$



Multicycle Boris
solver

$$\propto \left(\frac{\Delta t}{n} \right)^2$$

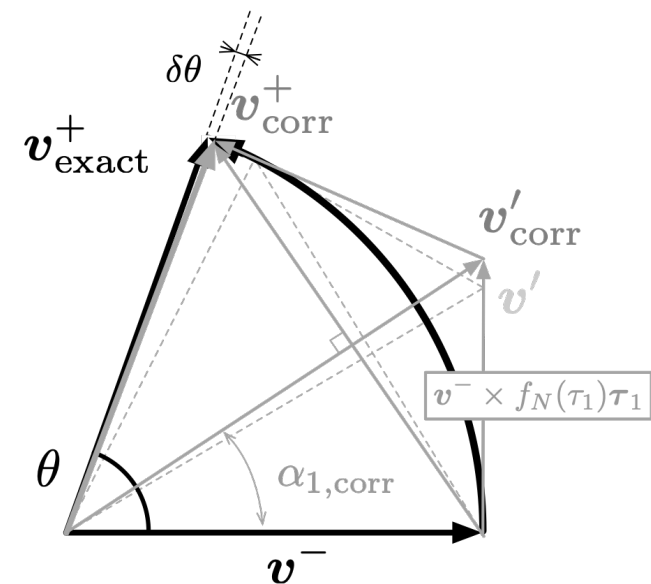
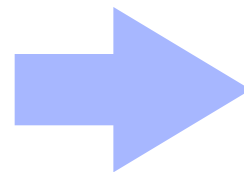
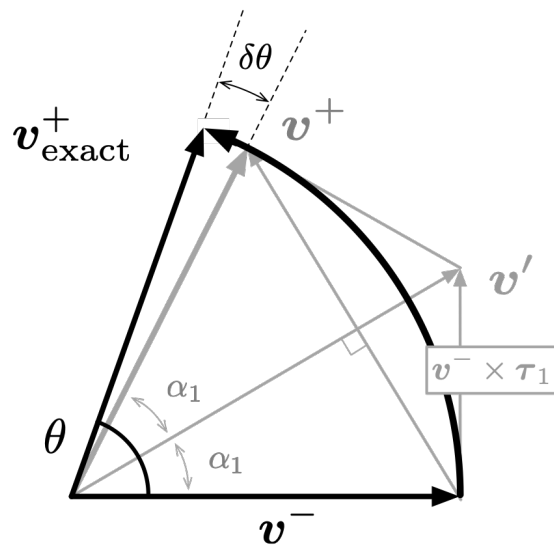
Solution 2 – Higher-order correction

- Boris (1970) has proposed a correction to the gyration part

element B vector $t_1 \equiv f_1 \frac{q\Delta t}{2m} B$

gyrophase correction $t_1 \leftarrow f_1 t_1$

$$f_1 \equiv \frac{\tan t_1}{t_1} = 1 + \frac{1}{3}t_1^2 + \frac{2}{15}t_1^4 + \frac{17}{315}t_1^6 + \frac{62}{2835}t_1^8 + \dots$$



Anisotropic correction to E field

- Buneman-Boris solver approx. angle θ_1

$$\left(\mathbf{v}^{(1)} - \frac{\mathbf{E} \times \mathbf{B}}{B^2} \right) = \left(\mathbf{v}^{(0)} - \frac{\mathbf{E} \times \mathbf{B}}{B^2} \right) \cos \theta_1 + \left(\left(\mathbf{v}^{(0)} - \frac{\mathbf{E} \times \mathbf{B}}{B^2} \right) \times \hat{\mathbf{b}} \right) \sin \theta_1 + \left(\mathbf{v}^{(0)} - \frac{\mathbf{E} \times \mathbf{B}}{B^2} \right)_{\parallel} (1 - \cos \theta_1) + \frac{q \mathbf{E}_{\parallel}}{m} \Delta t$$

Buneman 1967

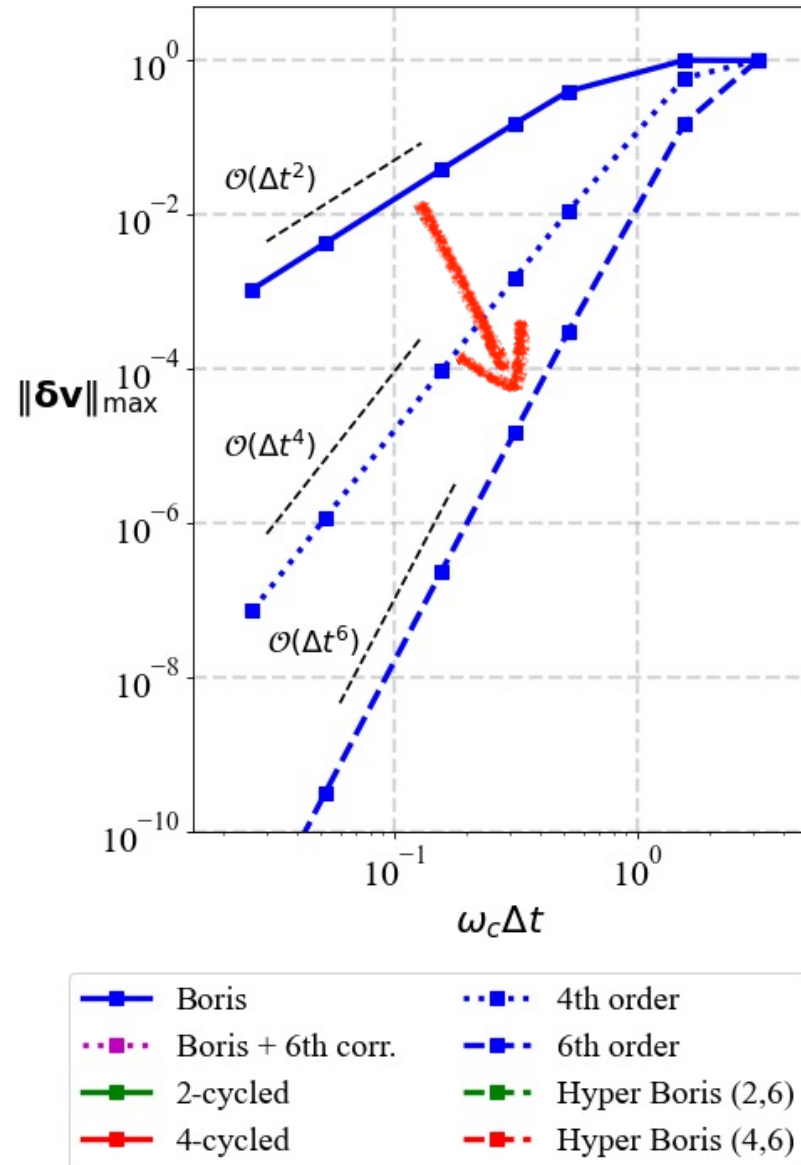
- Using a correction factor f_1 ...

$$\left(\mathbf{v}^{(1)} - \frac{\mathbf{E} \times \mathbf{B}}{f_1 B^2} \right) = \left(\mathbf{v}^{(0)} - \frac{\mathbf{E} \times \mathbf{B}}{f_1 B^2} \right) \cos \theta_1^{\text{corr}} + \left(\left(\mathbf{v}^{(0)} - \frac{\mathbf{E} \times \mathbf{B}}{f_1 B^2} \right) \times \hat{\mathbf{b}} \right) \sin \theta_1^{\text{corr}} + \left(\mathbf{v}^{(0)} - \frac{\mathbf{E} \times \mathbf{B}}{f_1 B^2} \right)_{\parallel} (1 - \cos \theta_1^{\text{corr}}) + \frac{q \mathbf{E}_{\parallel}}{m} \Delta t$$

One can eliminate the 2nd-order error by amplifying $E_{\perp}$ but by keeping $E_{\parallel}$

$$\boldsymbol{\varepsilon}_1 \equiv f_1 \mathbf{e}_1 + (1 - f_1) \mathbf{e}_{1,\parallel} = f_1 \mathbf{e}_1 + \frac{1 - f_1}{t_1^2} (\mathbf{e}_1 \cdot \mathbf{t}_1) \mathbf{t}_1$$

Numerical test: higher-order correction



- In addition to the correction to B, we have applied the anisotropic correction to E
- We obtain higher-order results

$$\propto (\Delta t)^N$$

Solution 3 — Hybrid method

- 0. Buneman-Boris solver $\propto (\Delta t)^2$



- 1. Subcycling

- It virtually **repeats** the 4-step procedure

$$\propto \left(\frac{\Delta t}{n}\right)^2$$

- 2. Higher-order correction

- Modification **before** the 4-step procedure

$$\propto (\Delta t)^N$$

- 3. Hyper Boris solver

- n-times cycling & Nth-order correction

Sample Recipe: (4_{cycle},6th-order) solver

- 1. Define the element vectors

$$\boldsymbol{\tau}_4 \equiv \frac{q\Delta t}{8m} \mathbf{B} \quad \boldsymbol{\varepsilon}_4 \equiv \frac{q\Delta t}{8m} \mathbf{E}$$

- 2. Higher-order correction

$$\mathbf{t}_4 \equiv \left(1 + \frac{1}{3}\tau_4^2 + \frac{2}{15}\tau_4^4\right) \boldsymbol{\tau}_4 \quad \mathbf{e}_4 \equiv \left(1 + \frac{1}{3}\tau_4^2 + \frac{2}{15}\tau_4^4\right) \boldsymbol{\varepsilon}_4 - \left(\frac{1}{3} + \frac{2}{15}\tau_4^2\right) (\boldsymbol{\varepsilon}_4 \cdot \boldsymbol{\tau}_4) \boldsymbol{\tau}_4$$

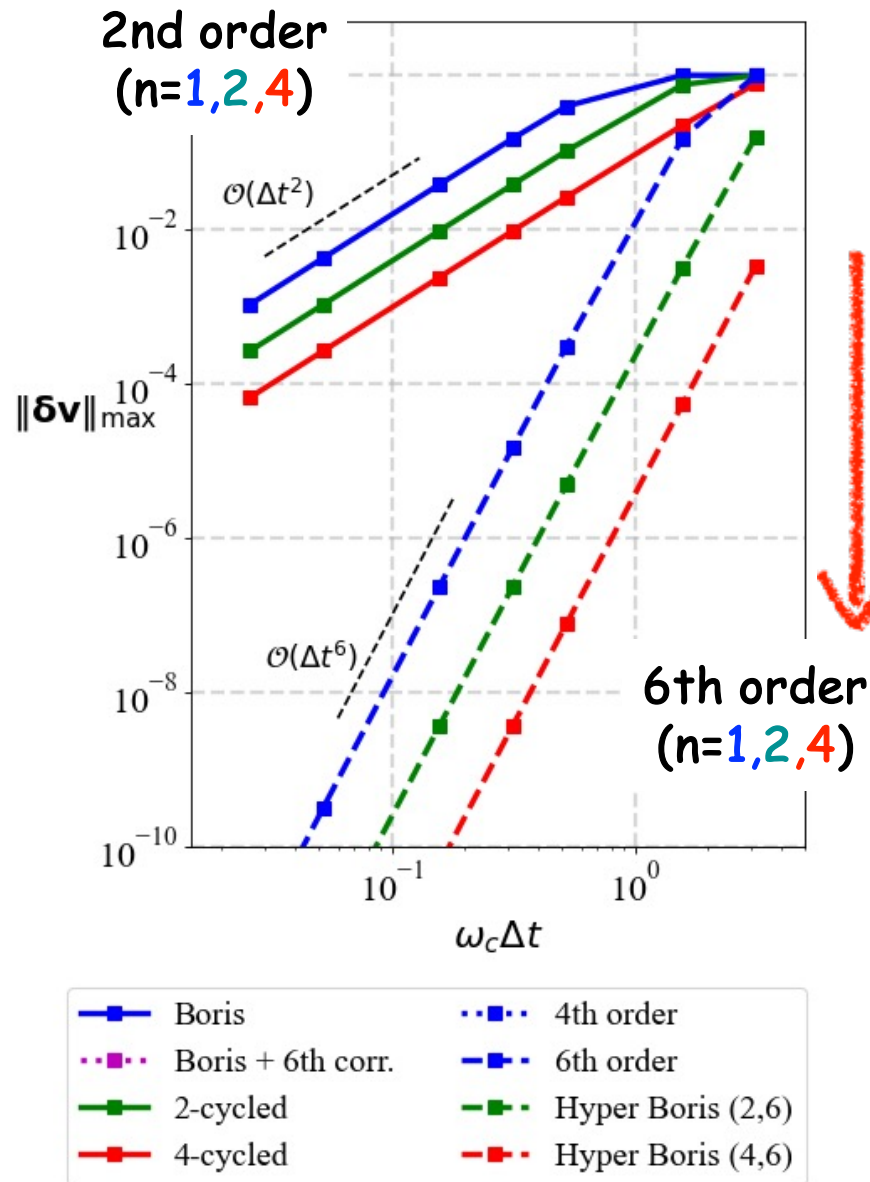
- 3. Calculate the coefficients

$$c_{41} = \frac{1 - 28t_4^2 + 70t_4^4 - 28t_4^6 + t_4^8}{(1 + t_4^2)^4}, \quad c_{42} = \frac{8(1 - 7t_4^2 + 7t_4^4 - t_4^6)}{(1 + t_4^2)^4}, \quad c_{43} = \frac{32(1 - t_4^2)^2}{(1 + t_4^2)^4},$$
$$c_{44} = c_{42}, \quad c_{45} = c_{43}, \quad c_{46} = \frac{8(11 - t_4^2 + 5t_4^4 + t_4^6)}{(1 + t_4^2)^4}$$

- 4. Calculate the velocity at the next timestep

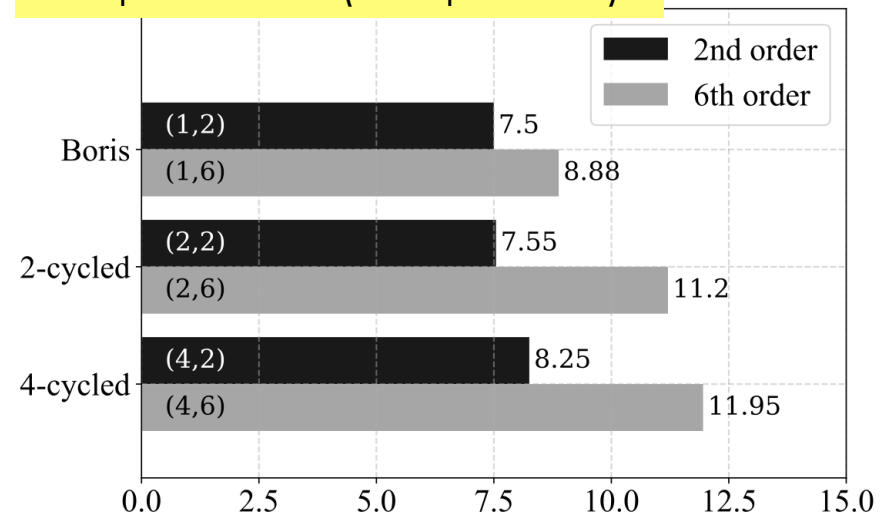
$$\begin{aligned} \mathbf{v}^{t+\Delta t} = & c_{41} \mathbf{v}^t + c_{42} (\mathbf{v}^t \times \mathbf{t}_4) + c_{43} (\mathbf{v}^t \cdot \mathbf{t}_4) \mathbf{t}_4 \\ & + c_{44} \mathbf{e}_4 + c_{45} (\mathbf{e}_4 \times \mathbf{t}_4) + c_{46} (\mathbf{e}_4 \cdot \mathbf{t}_4) \mathbf{t}_4 \end{aligned}$$

Hyper Boris solver: Numerical tests (1)



- Numerical accuracy of $\propto \left(\frac{\Delta t}{n}\right)^N$
- Affordable computational costs

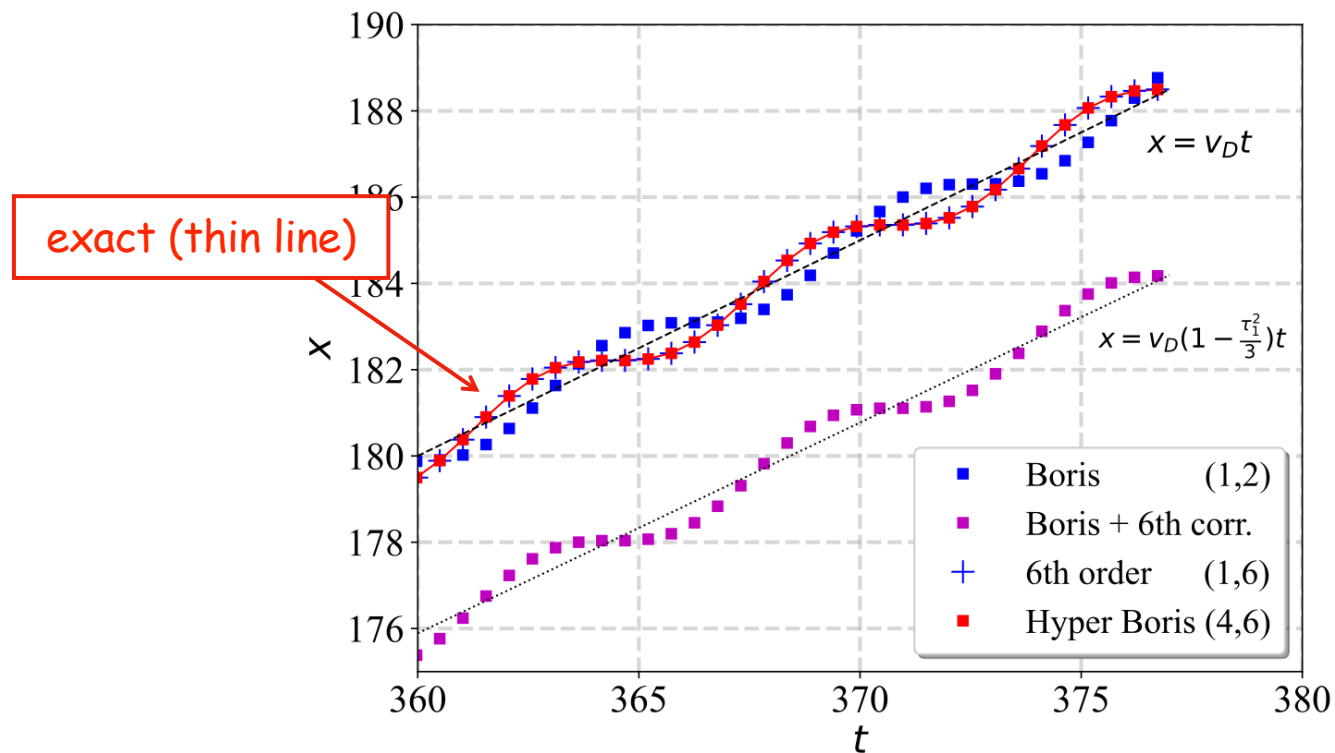
Elapsed time (test-particle)



Hyper Boris solver: Numerical tests (2)

- Long-term ExB drift

- Popular gyrophase correction slows down the drift speed
- Our correction preserves the drift speed



Summary

- 0. Buneman-Boris solver $\propto (\Delta t)^2$



- 1. Subcycling

- Multicycle formula with Chebyshev polynomials for arbitrary n

$$\propto \left(\frac{\Delta t}{n}\right)^2$$

- 2. Higher-order correction

- Anisotropic correction to the electric field

$$\propto (\Delta t)^N$$



- 3. Hyper Boris solver

- n-times cycling & Nth-order cor.
- Applicable to any problem of

$$\propto \left(\frac{\Delta t}{n}\right)^N$$
$$\frac{d\mathbf{v}}{dt} = \mathbf{F} + \mathbf{v} \times \mathbf{R}$$

Better simulations for kinetic plasma problems

Frequently Asked Questions

- 1. Is the hyper Boris solver symplectic?
 - No. Boris-type solvers are not symplectic.
- 2. Does it have a long-term stability?
 - Yes. Because it preserves the phase-space volume.
- 3. What hyperparameters do you recommend?
 - I recommend (2-cycle, 6th-order) or (4-cycle, 6th) solvers.
- 4. Can the hyper Boris solver solve relativistic motion?
 - No. We can actually use multicycling, but we can no longer use the multicycle formula.
- 5. What should we use in the relativistic regime?
 - For a moment, I recommend the Vay solver (Vay 2008) or the quadruple Boris solver (Zenitani & Kato 2020).
 - We need to develop new solvers.

Coefficients (numerators only)

c11: $1 - t^{**2}$

c12: 2

c13: 2

c14: 2

c15: 2

c16: 2

c21: $t^{**4} - 6*t^{**2} + 1$

c22: $4 - 4*t^{**2}$

c23: 8

c24: $4 - 4*t^{**2}$

c25: 8

c26: $4*t^{**2} + 12$

c31: $-t^{**6} + 15*t^{**4} - 15*t^{**2} + 1$

c32: $6*t^{**4} - 20*t^{**2} + 6$

c33: $-2*t^{**4} + 4*t^{**2} + 6$

c34: $6*t^{**4} - 20*t^{**2} + 6$

c35: $-2*t^{**4} + 4*t^{**2} + 6$

c36: $6*t^{**4} + 12*t^{**2} + 38$

c41: $t^{**8} - 28*t^{**6} + 70*t^{**4} - 28*t^{**2} + 1$

c42: $-8*t^{**6} + 56*t^{**4} - 56*t^{**2} + 8$

c43: $32*t^{**4} - 64*t^{**2} + 32$

c44: $-8*t^{**6} + 56*t^{**4} - 56*t^{**2} + 8$

c45: $32*t^{**4} - 64*t^{**2} + 32$

c46: $8*t^{**6} + 40*t^{**4} - 8*t^{**2} + 88$

c51: $-t^{**10} + 45*t^{**8} - 210*t^{**6} + 210*t^{**4} - 45*t^{**2} + 1$

c52: $10*t^{**8} - 120*t^{**6} + 252*t^{**4} - 120*t^{**2} + 10$

c53: $2*t^{**8} - 16*t^{**6} - 28*t^{**4} + 10$

c54: $10*t^{**8} - 120*t^{**6} + 252*t^{**4} - 120*t^{**2} + 10$

c55: $2*t^{**8} - 16*t^{**6} - 28*t^{**4} + 10$

c56: $10*t^{**8} + 40*t^{**6} + 220*t^{**4} - 152*t^{**2} + 170$

c61: $t^{**12} - 66*t^{**10} + 495*t^{**8} - 924*t^{**6} + 495*t^{**4} - 66*t^{**2} + 1$

c62: $-12*t^{**10} + 220*t^{**8} - 792*t^{**6} + 792*t^{**4} - 220*t^{**2} + 12$

c63: $72*t^{**8} - 480*t^{**6} + 944*t^{**4} - 480*t^{**2} + 72$

c64: $-12*t^{**10} + 220*t^{**8} - 792*t^{**6} + 792*t^{**4} - 220*t^{**2} + 12$

c65: $72*t^{**8} - 480*t^{**6} + 944*t^{**4} - 480*t^{**2} + 72$

c66: $12*t^{**10} - 84*t^{**8} + 480*t^{**6} - 480*t^{**4} + 612*t^{**2} - 282$

Coefficients (numerators only)

c71: -t**14 + 91*t**12 - 1001*t**10 + 3003*t**8 - 3003*t**6 + 1001*t**4 - 91*t**2 + 1
c72: 14*t**12 - 364*t**10 + 2002*t**8 - 3432*t**6 + 2002*t**4 - 364*t**2 + 14
c73: -2*t**12 + 36*t**10 + 50*t**8 - 72*t**6 - 126*t**4 - 28*t**2 + 14
c74: 14*t**12 - 364*t**10 + 2002*t**8 - 3432*t**6 + 2002*t**4 - 364*t**2 + 14
c75: -2*t**12 + 36*t**10 + 50*t**8 - 72*t**6 - 126*t**4 - 28*t**2 + 14
c76: 14*t**12 + 84*t**10 + 658*t**8 - 1512*t**6 + 3922*t**4 - 1708*t**2 + 462

c81: t**16 - 120*t**14 + 1820*t**12 - 8008*t**10 + 12870*t**8 - 8008*t**6 + 1820*t**4 - 120*t**2 + 1
c82: -16*t**14 + 560*t**12 - 4368*t**10 + 11440*t**8 - 11440*t**6 + 4368*t**4 - 560*t**2 + 16
c83: 128*t**12 - 1792*t**10 + 8064*t**8 - 12800*t**6 + 8064*t**4 - 1792*t**2 + 128
c84: -16*t**14 + 560*t**12 - 4368*t**10 + 11440*t**8 - 11440*t**6 + 4368*t**4 - 560*t**2 + 16
c85: 128*t**12 - 1792*t**10 + 8064*t**8 - 12800*t**6 + 8064*t**4 - 1792*t**2 + 128
c86: 16*t**14 + 144*t**12 - 112*t**10 + 5264*t**8 - 10320*t**6 + 12336*t**4 - 3920*t**2 + 688

c91: -t**18 + 153*t**16 - 3060*t**14 + 18564*t**12 - 43758*t**10 + 43758*t**8 - 18564*t**6 + 3060*t**4 - 153*t**2 + 1
c92: 18*t**16 - 816*t**14 + 8568*t**12 - 31824*t**10 + 48620*t**8 - 31824*t**6 + 8568*t**4 - 816*t**2 + 18
c93: 2*t**16 - 64*t**14 - 24*t**12 + 416*t**10 + 572*t**8 - 312*t**6 - 96*t**4 + 18
c94: 18*t**16 - 816*t**14 + 8568*t**12 - 31824*t**10 + 48620*t**8 - 31824*t**6 + 8568*t**4 - 816*t**2 + 18
c95: 2*t**16 - 64*t**14 - 24*t**12 + 416*t**10 + 572*t**8 - 312*t**6 - 96*t**4 + 18
c96: 18*t**16 + 144*t**14 + 1464*t**12 - 7056*t**10 + 34092*t**8 - 46352*t**6 + 33336*t**4 - 7920*t**2 + 978

c101: t**20 - 190*t**18 + 4845*t**16 - 38760*t**14 + 125970*t**12 - 184756*t**10 + 125970*t**8 - 38760*t**6 + 4845*t**4 - 190*t**2 + 1
c102: -20*t**18 + 1140*t**16 - 15504*t**14 + 77520*t**12 - 167960*t**10 + 167960*t**8 - 77520*t**6 + 15504*t**4 - 1140*t**2 + 20
c103: 200*t**16 - 4800*t**14 + 38880*t**12 - 125760*t**10 + 185008*t**8 - 125760*t**6 + 38880*t**4 - 4800*t**2 + 200
c104: -20*t**18 + 1140*t**16 - 15504*t**14 + 77520*t**12 - 167960*t**10 + 167960*t**8 - 77520*t**6 + 15504*t**4 - 1140*t**2 + 20
c105: 200*t**16 - 4800*t**14 + 38880*t**12 - 125760*t**10 + 185008*t**8 - 125760*t**6 + 38880*t**4 - 4800*t**2 + 200
c106: 20*t**18 + 220*t**16 - 240*t**14 + 17904*t**12 - 73320*t**10 + 173008*t**8 - 163760*t**6 + 79920*t**4 - 14604*t**2 + 1340

c111: -t**22 + 231*t**20 - 7315*t**18 + 74613*t**16 - 319770*t**14 + 646646*t**12 - 646646*t**10 + 319770*t**8 - 74613*t**6 + 7315*t**4 - 231*t**2 + 1
c112: 22*t**20 - 1540*t**18 + 26334*t**16 - 170544*t**14 + 497420*t**12 - 705432*t**10 + 497420*t**8 - 170544*t**6 + 26334*t**4 - 1540*t**2 + 22
c113: -2*t**20 + 100*t**18 - 130*t**16 - 1296*t**14 - 1220*t**12 + 1560*t**10 + 2860*t**8 + 880*t**6 - 506*t**4 - 220*t**2 + 22
c114: 22*t**20 - 1540*t**18 + 26334*t**16 - 170544*t**14 + 497420*t**12 - 705432*t**10 + 497420*t**8 - 170544*t**6 + 26334*t**4 - 1540*t**2 + 22
c115: -2*t**20 + 100*t**18 - 130*t**16 - 1296*t**14 - 1220*t**12 + 1560*t**10 + 2860*t**8 + 880*t**6 - 506*t**4 - 220*t**2 + 22
c116: 22*t**20 + 220*t**18 + 2750*t**16 - 22704*t**14 + 177804*t**12 - 487256*t**10 + 715596*t**8 - 490160*t**6 + 174174*t**4 - 25124*t**2 + 1782

c121: t**24 - 276*t**22 + 10626*t**20 - 134596*t**18 + 735471*t**16 - 1961256*t**14 + 2704156*t**12 - 1961256*t**10 + 735471*t**8 - 134596*t**6 + 10626*t**4 - 276*t**2 + 1
c122: -24*t**22 + 2024*t**20 - 42504*t**18 + 346104*t**16 - 1307504*t**14 + 2496144*t**12 - 2496144*t**10 + 1307504*t**8 - 346104*t**6 + 42504*t**4 - 2024*t**2 + 24
c123: 288*t**20 - 10560*t**18 + 134816*t**16 - 734976*t**14 + 1962048*t**12 - 2703232*t**10 + 1962048*t**8 - 734976*t**6 + 134816*t**4 - 10560*t**2 + 288
c124: -24*t**22 + 2024*t**20 - 42504*t**18 + 346104*t**16 - 1307504*t**14 + 2496144*t**12 - 2496144*t**10 + 1307504*t**8 - 346104*t**6 + 42504*t**4 - 2024*t**2 + 24
c125: 288*t**20 - 10560*t**18 + 134816*t**16 - 734976*t**14 + 1962048*t**12 - 2703232*t**10 + 1962048*t**8 - 734976*t**6 + 134816*t**4 - 10560*t**2 + 288
c126: 24*t**22 + 312*t**20 - 440*t**18 + 47784*t**16 - 334224*t**14 + 1326512*t**12 - 2473968*t**10 + 2515152*t**8 - 1295624*t**6 + 351384*t**4 - 40920*t**2 + 2312

c131: -t**26 + 325*t**24 - 14950*t**22 + 230230*t**20 - 1562275*t**18 + 5311735*t**16 - 9657700*t**14 + 9657700*t**12 - 5311735*t**10 + 1562275*t**8 - 230230*t**6 + 14950*t**4 - 325*t**2 + 1
c132: 26*t**24 - 2600*t**22 + 65780*t**20 - 657800*t**18 + 3124550*t**16 - 7726160*t**14 + 10400600*t**12 - 7726160*t**10 + 3124550*t**8 - 657800*t**6 + 65780*t**4 - 2600*t**2 + 26
c133: 2*t**24 - 144*t**22 + 524*t**20 + 2848*t**18 + 342*t**16 - 10336*t**14 - 12920*t**12 + 8398*t**8 + 3952*t**6 - 468*t**4 - 416*t**2 + 26
c134: 26*t**24 - 2600*t**22 + 65780*t**20 - 657800*t**18 + 3124550*t**16 - 7726160*t**14 + 10400600*t**12 - 7726160*t**10 + 3124550*t**8 - 657800*t**6 + 65780*t**4 - 2600*t**2 + 26
c135: 2*t**24 - 144*t**22 + 524*t**20 + 2848*t**18 + 342*t**16 - 10336*t**14 - 12920*t**12 + 8398*t**8 + 3952*t**6 - 468*t**4 - 416*t**2 + 26
c136: 26*t**24 + 312*t**22 + 4628*t**20 - 58344*t**18 + 676390*t**16 - 3091088*t**14 + 7770776*t**12 - 10355984*t**10 + 7759622*t**8 - 3105960*t**6 + 665236*t**4 - 63752*t**2 + 2938

c141: t**28 - 378*t**26 + 20475*t**24 - 376740*t**22 + 3108105*t**20 - 13123110*t**18 + 30421755*t**16 - 40116600*t**14 + 30421755*t**12 - 13123110*t**10 + 3108105*t**8 - 376740*t**6 + 20475*t**4 - 378*t**2 + 1
c142: -28*t**26 + 3276*t**24 - 98280*t**22 + 1184040*t**20 - 6906900*t**18 + 21474180*t**16 - 37442160*t**14 + 37442160*t**12 - 21474180*t**10 + 6906900*t**8 - 1184040*t**6 + 98280*t**4 - 3276*t**2 + 28
c143: 392*t**24 - 20384*t**22 + 377104*t**20 - 3107104*t**18 + 13125112*t**16 - 30418752*t**14 + 40120032*t**12 - 30418752*t**10 + 13125112*t**8 - 3107104*t**6 + 377104*t**4 - 20384*t**2 + 392
c144: -28*t**26 + 3276*t**24 - 98280*t**22 + 1184040*t**20 - 6906900*t**18 + 21474180*t**16 - 37442160*t**14 + 37442160*t**12 - 21474180*t**10 + 6906900*t**8 - 1184040*t**6 + 98280*t**4 - 3276*t**2 + 28
c145: 392*t**24 - 20384*t**22 + 377104*t**20 - 3107104*t**18 + 13125112*t**16 - 30418752*t**14 + 40120032*t**12 - 30418752*t**10 + 13125112*t**8 - 3107104*t**6 + 377104*t**4 - 20384*t**2 + 392
c146: 28*t**26 + 420*t**24 - 728*t**22 + 108472*t**20 - 1156012*t**18 + 6962956*t**16 - 21390096*t**14 + 37538256*t**12 - 37358076*t**10 + 21530236*t**8 - 6878872*t**6 + 1194232*t**4 - 95732*t**2 + 3668

c151: -t**30 + 435*t**28 - 27405*t**26 + 593775*t**24 - 5852925*t**22 + 30045015*t**20 - 86493225*t**18 + 145422675*t**16 - 145422675*t**14 + 86493225*t**12 - 30045015*t**10 + 5852925*t**8 - 593775*t**6 + 27405*t**4 - 435*t**2 + 1
c152: 30*t**28 - 4060*t**26 + 142506*t**24 - 2035800*t**22 + 14307150*t**20 - 54627300*t**18 + 119759850*t**16 - 155117520*t**14 + 119759850*t**12 - 54627300*t**10 + 14307150*t**8 - 2035800*t**6 + 142506*t**4 - 4060*t**2 + 30
c153: -2*t**28 + 196*t**26 - 1302*t**24 - 4760*t**22 + 7150*t**20 + 37884*t**18 + 30058*t**16 - 36176*t**14 - 67830*t**12 - 23940*t**10 + 15246*t**8 + 11112*t**6 + 266*t**4 - 700*t**2 + 30
c154: 30*t**28 - 4060*t**26 + 142506*t**24 - 2035800*t**22 + 14307150*t**20 - 54627300*t**18 + 119759850*t**16 - 155117520*t**14 + 119759850*t**12 - 54627300*t**10 + 14307150*t**8 - 2035800*t**6 + 142506*t**4 - 4060*t**2 + 30
c155: -2*t**28 + 196*t**26 - 1302*t**24 - 4760*t**22 + 7150*t**20 + 37884*t**18 + 30058*t**16 - 36176*t**14 - 67830*t**12 - 23940*t**10 + 15246*t**8 + 11112*t**6 + 266*t**4 - 700*t**2 + 30
c156: 30*t**28 + 420*t**26 + 7210*t**24 - 128856*t**22 + 2076750*t**20 - 14217060*t**18 + 54777450*t**16 - 119566800*t**14 + 155310570*t**12 - 119609700*t**10 + 54717390*t**8 - 14266200*t**6 + 2049450*t**4 - 139356*t**2 + 4510

c161: t**32 - 496*t**30 + 35960*t**28 - 906192*t**26 + 10518300*t**24 - 64512240*t**22 + 225792840*t**20 - 471435600*t**18 + 601080390*t**16 - 471435600*t**14 + 225792840*t**12 - 64512240*t**10 + 10518300*t**8 - 906192*t**6 + 35960*t**4 - 496*t**2 + 1
c162: -32*t**30 + 4960*t**28 - 201376*t**26 + 3365856*t**24 - 28048800*t**22 + 129024480*t**20 - 347373600*t**18 + 565722720*t**16 - 565722720*t**14 + 347373600*t**12 - 129024480*t**10 + 28048800*t**8 - 3365856*t**6 + 201376*t**4 - 4960*t**2 + 32
c163: 512*t**28 - 35840*t**26 + 906752*t**24 - 10516480*t**22 + 64516608*t**20 - 225784832*t**18 + 471447040*t**16 - 601067520*t**14 + 471447040*t**12 - 225784832*t**10 + 64516608*t**8 - 10516480*t**6 + 906752*t**4 - 35840*t**2 + 512
c164: -32*t**30 + 4960*t**28 - 201376*t**26 + 3365856*t**24 - 28048800*t**22 + 129024480*t**20 - 347373600*t**18 + 565722720*t**16 - 565722720*t**14 + 347373600*t**12 - 129024480*t**10 + 28048800*t**8 - 3365856*t**6 + 201376*t**4 - 4960*t**2 + 32
c165: 512*t**28 - 35840*t**26 + 906752*t**24 - 10516480*t**22 + 64516608*t**20 - 225784832*t**18 + 471447040*t**16 - 601067520*t**14 + 471447040*t**12 - 225784832*t**10 + 64516608*t**8 - 10516480*t**6 + 906752*t**4 - 35840*t**2 + 512
c166: 32*t**30 + 544*t**28 - 1120*t**26 + 219296*t**24 - 3307616*t**22 + 28188576*t**20 - 128768224*t**18 + 347739680*t**16 - 565310880*t**14 + 566088800*t**12 - 347117344*t**10 + 129164256*t**8 - 27990560*t**6 + 3383776*t**4 - 197536*t**2 + 5472