ÖAW AUSTRIAN ACADEMY OF SCIENCES

IWF SPACE RESEARCH INSTITUTE

Hyper Boris integrators for particle-in-cell simulation

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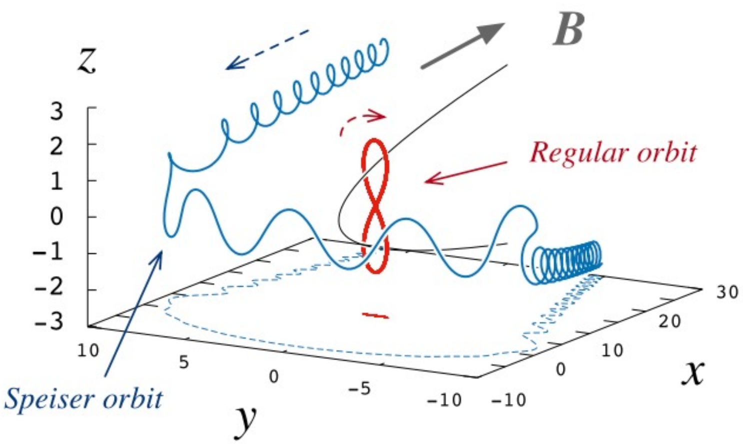
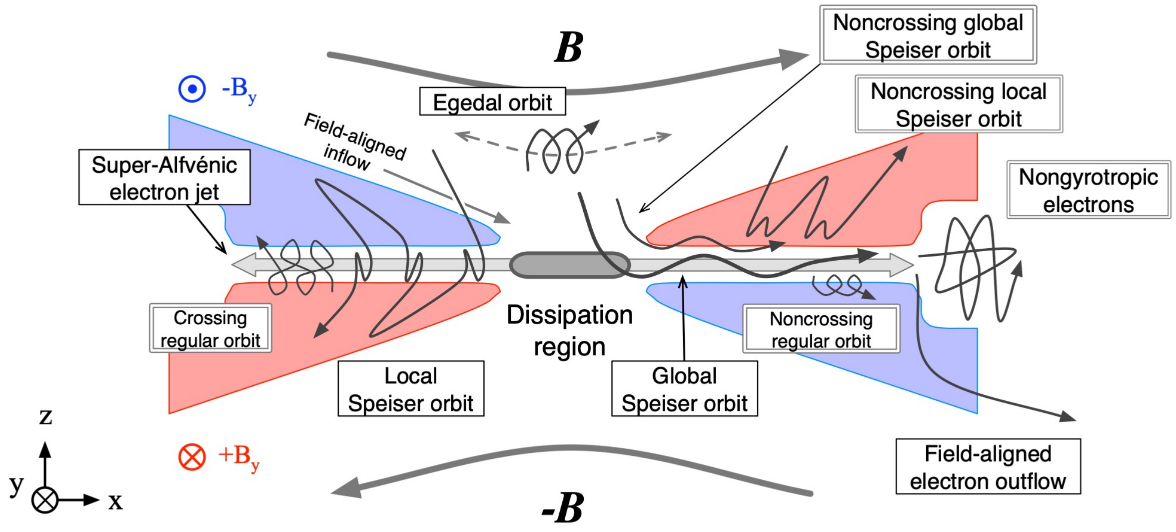
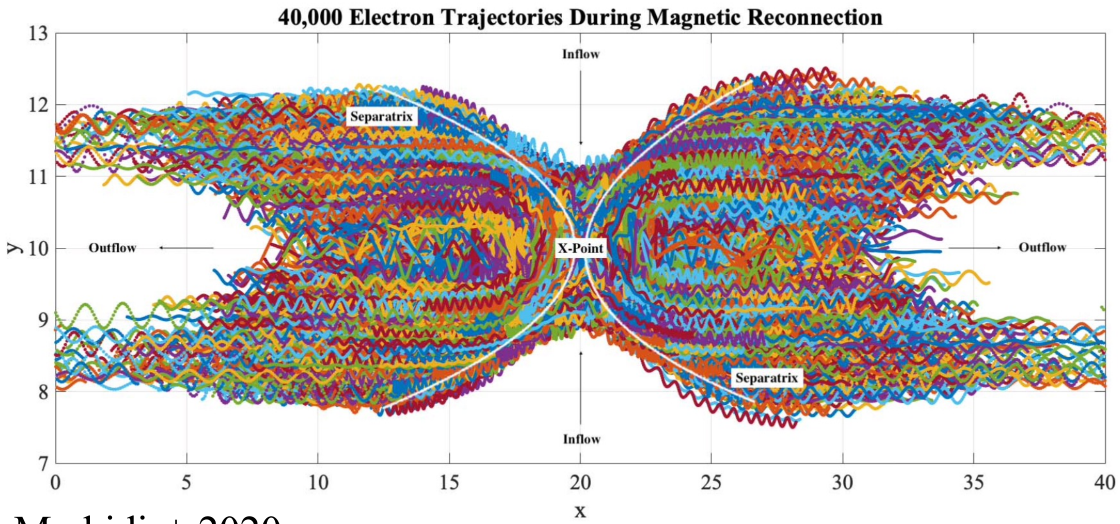
Particle motion in reconnection





Zenitani & Nagai 2016



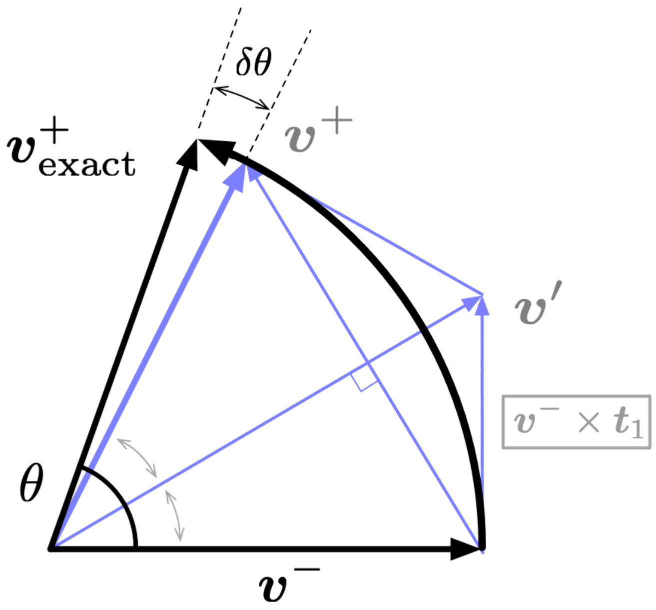
Markidis+ 2020

It is important to accuretely solve particle motion in PIC simulation

Boris solver (a.k.a. Buneman-Boris solver)

$$\frac{\boldsymbol{x}^{t+\Delta t}-\boldsymbol{x}^{t}}{\Delta t}=\boldsymbol{v}^{t+\frac{1}{2}\Delta t}$$

$$m\,\frac{\boldsymbol{v}^{t+\frac{1}{2}\Delta t}-\boldsymbol{v}^{t-\frac{1}{2}\Delta t}}{\Delta t}=q\left(\boldsymbol{E}^{t}+\frac{\boldsymbol{v}^{t+\frac{1}{2}\Delta t}+\boldsymbol{v}^{t-\frac{1}{2}\Delta t}}{2}\times\boldsymbol{B}^{t}\right)$$



element E vector   element B vector

$$\boldsymbol{e}_1\equiv\frac{q\Delta t}{2m}\boldsymbol{E},\quad \boldsymbol{t}_1\equiv\frac{q\Delta t}{2m}\boldsymbol{B}$$

$$\boldsymbol{v}^{-}=\boldsymbol{v}^{t-\frac{1}{2}\Delta t}+\boldsymbol{e}_1$$ Half acc. by E

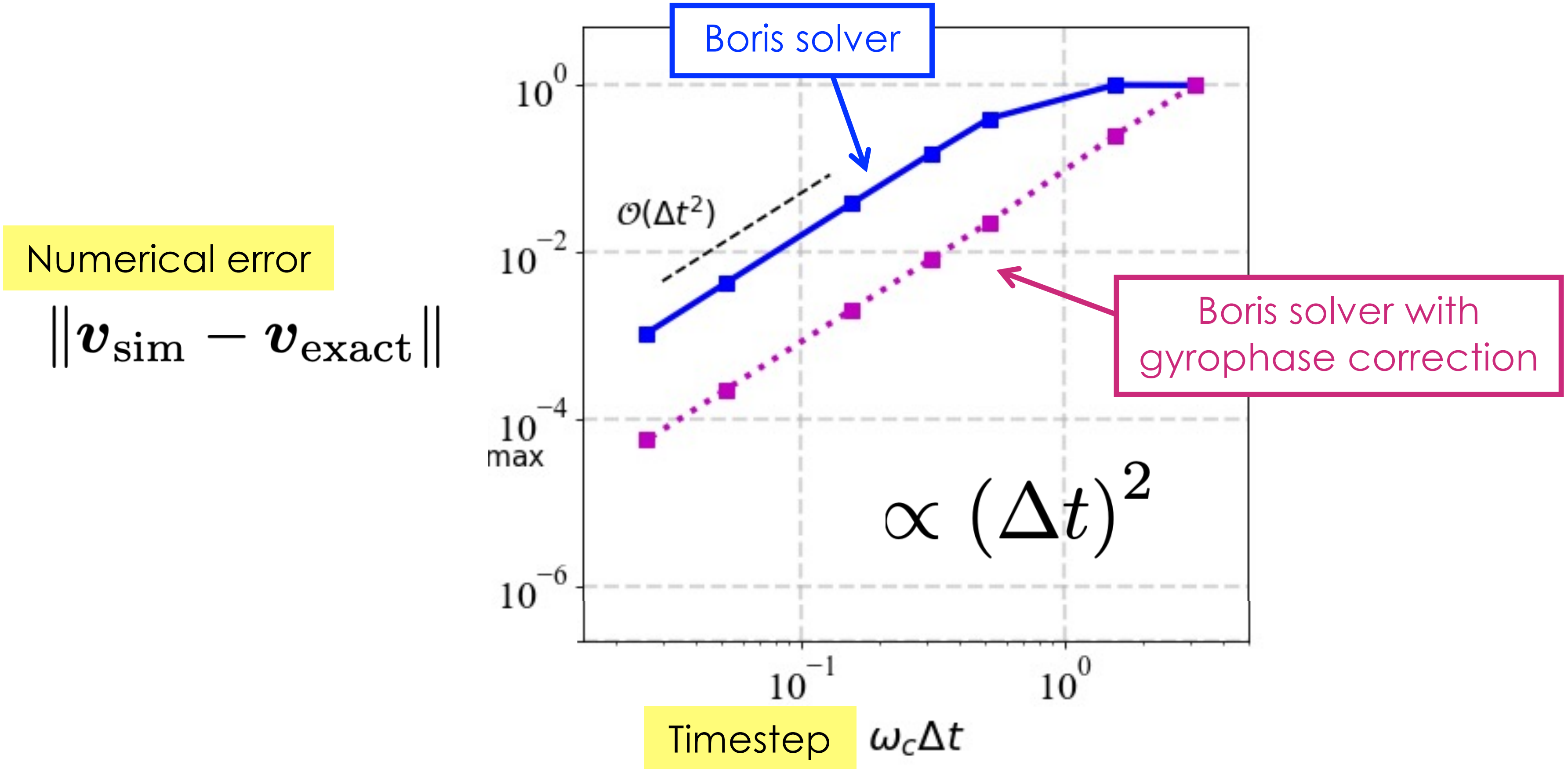
$$\boldsymbol{v}'=\boldsymbol{v}^{-}+\boldsymbol{v}^{-}\times\boldsymbol{t}_1$$

$$\boldsymbol{v}^{+}=\boldsymbol{v}^{-}+\frac{2}{1+t_1^2}\,\boldsymbol{v}'\times\boldsymbol{t}_1$$ Approximate gyration

$$\boldsymbol{v}^{t+\frac{1}{2}\Delta t}=\boldsymbol{v}^{+}+\boldsymbol{e}_1$$ Half acc. by E

Boris 1970

Boris solver: 2nd-order accuracy



We propose 3 solutions to improve the Boris solver

Solution 1: Subcycling

- We repeat the procedure n times with $\frac{\Delta t}{n}$
- $\boldsymbol{E}$ & $\boldsymbol{B}$ are fixed

element E vector element B vector

$$\boldsymbol{e}_1 \equiv \frac{q\Delta t}{2m}\boldsymbol{E}, \quad \boldsymbol{t}_1 \equiv \frac{q\Delta t}{2m}\boldsymbol{B}$$

$$\begin{aligned} \boldsymbol{v}^- &= \boldsymbol{v}^{t-\frac{1}{2}\Delta t} + \boldsymbol{e}_1 \\ \boldsymbol{v}' &= \boldsymbol{v}^- + \boldsymbol{v}^- \times \boldsymbol{t}_1 \\ \boldsymbol{v}^+ &= \boldsymbol{v}^- + \frac{2}{1+t_1^2}\boldsymbol{v}' \times \boldsymbol{t}_1 \\ \boldsymbol{v}^{t+\frac{1}{2}\Delta t} &= \boldsymbol{v}^+ + \boldsymbol{e}_1 \end{aligned}$$

subcycling →

element E vector element B vector

$$\boldsymbol{e}_n \equiv \frac{q\Delta t}{2nm}\boldsymbol{E}, \quad \boldsymbol{t}_n \equiv \frac{q\Delta t}{2nm}\boldsymbol{B}$$

$$\left.\begin{matrix} \begin{cases} \boldsymbol{v}^- = \boldsymbol{v}^{(0)} + \boldsymbol{e}_n \\ \boldsymbol{v}' = \boldsymbol{v}^- + \boldsymbol{v}^- \times \boldsymbol{t}_n \\ \boldsymbol{v}^+ = \boldsymbol{v}^- + \frac{2}{1+t_n^2}\boldsymbol{v}' \times \boldsymbol{t}_n \\ \boldsymbol{v}^{(1)} = \boldsymbol{v}^+ + \boldsymbol{e}_n \end{cases} \\ \vdots \\ \begin{cases} \boldsymbol{v}^- = \boldsymbol{v}^{(n-1)} + \boldsymbol{e}_n \\ \boldsymbol{v}' = \boldsymbol{v}^- + \boldsymbol{v}^- \times \boldsymbol{t}_n \\ \boldsymbol{v}^+ = \boldsymbol{v}^- + \frac{2}{1+t_n^2}\boldsymbol{v}' \times \boldsymbol{t}_n \\ \boldsymbol{v}^{(n)} = \boldsymbol{v}^+ + \boldsymbol{e}_n \end{cases} \end{matrix}\right\} \times n \text{ times}$$

Multicycle formula for arbitrary n

1-cycle

$$\boldsymbol{v}^{t+\frac{1}{2}\Delta t} = \frac{1-t_1^2}{1+t_1^2}\boldsymbol{v}^{t-\frac{1}{2}\Delta t} + \frac{2}{1+t_1^2}\left(\boldsymbol{v}^{t-\frac{1}{2}\Delta t}\times \boldsymbol{t}_1\right) + \frac{2}{1+t_1^2}(\boldsymbol{v}^{t-\frac{1}{2}\Delta t}\cdot \boldsymbol{t}_1)\boldsymbol{t}_1$$
$$+ \frac{2}{1+t_1^2}\boldsymbol{e}_1 + \frac{2}{1+t_1^2}\left(\boldsymbol{e}_1\times \boldsymbol{t}_1\right) + \frac{2}{1+t_1^2}(\boldsymbol{e}_n\cdot \boldsymbol{t}_1)\boldsymbol{t}_1$$

2-cycles

$$\boldsymbol{v}^{t+\frac{1}{2}\Delta t} = \frac{1-6t_2^2+t_2^4}{(1+t_2^2)^2}\boldsymbol{v}^{t-\frac{1}{2}\Delta t} + \frac{4(1-t_2^2)}{(1+t_2^2)^2}\left(\boldsymbol{v}^{t-\frac{1}{2}\Delta t}\times \boldsymbol{t}_2\right) + \frac{8}{(1+t_2^2)^2}(\boldsymbol{v}^{t-\frac{1}{2}\Delta t}\cdot \boldsymbol{t}_2)\boldsymbol{t}_2$$
$$+ \frac{4(1-t_2^2)}{(1+t_2^2)^2}\boldsymbol{e}_2 + \frac{8}{(1+t_2^2)^2}\left(\boldsymbol{e}_2\times \boldsymbol{t}_2\right) + \frac{4(3+t_2^2)}{(1+t_2^2)^2}(\boldsymbol{e}_2\cdot \boldsymbol{t}_2)\boldsymbol{t}_2$$

n-cycles

$$\boldsymbol{v}^{t+\Delta t} = c_{n1}\boldsymbol{v}^t + c_{n2}\left(\boldsymbol{v}^t\times \boldsymbol{t}_n\right) + c_{n3}(\boldsymbol{v}^t\cdot \boldsymbol{t}_n)\boldsymbol{t}_n$$
$$+ c_{n4}\boldsymbol{e}_n + c_{n5}(\boldsymbol{e}_n\times \boldsymbol{t}_n) + c_{n6}(\boldsymbol{e}_n\cdot \boldsymbol{t}_n)\boldsymbol{t}_n$$

element E & B vectors

$$\boldsymbol{e}_n \equiv \frac{q\Delta t}{2nm}\boldsymbol{E}$$
$$\boldsymbol{t}_n \equiv \frac{q\Delta t}{2nm}\boldsymbol{B}$$

$$c_{n1} = T_n\left(\frac{1-t_n^2}{1+t_n^2}\right)$$
$$c_{n2} = c_{n4} = \frac{2}{1+t_n^2}U_{n-1}\left(\frac{1-t_n^2}{1+t_n^2}\right)$$
$$c_{n6} = \frac{2}{t_n^2}\left(n - \frac{1}{1+t_n^2}U_{n-1}\left(\frac{1-t_n^2}{1+t_n^2}\right)\right)$$

$$c_{n3} = c_{n5} = \begin{cases} \frac{2}{1+t_n^2} & (\text{for } n=1) \\ \frac{2}{1+t_n^2}\left(U_k\left(\frac{1-t_n^2}{1+t_n^2}\right) + U_{k-1}\left(\frac{1-t_n^2}{1+t_n^2}\right)\right)^2 & (\text{for } n=2\text{k}+1) \\ \frac{8}{(1+t_n^2)^2}\left(U_{k-1}\left(\frac{1-t_n^2}{1+t_n^2}\right)\right)^2 & (\text{for } n=2\text{k}) \end{cases}$$

T_n, U_n: Chebyshev polynomials

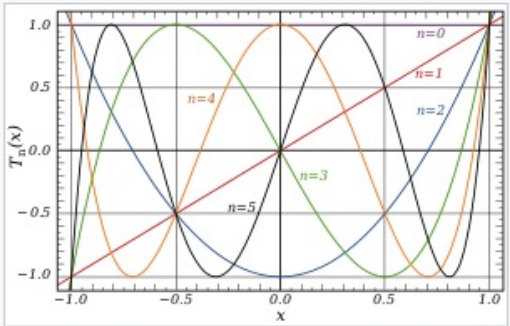
Zenitani & Kato 2020, 2025

Reminder: Chebyshev polynomials

First kind [edit]

The first few Chebyshev polynomials of the first kind are OEIS: A028297

$$\begin{aligned}
T_0(x) &= 1\\
T_1(x) &= x\\
T_2(x) &= 2x^2 - 1\\
T_3(x) &= 4x^3 - 3x\\
T_4(x) &= 8x^4 - 8x^2 + 1\\
T_5(x) &= 16x^5 - 20x^3 + 5x\\
T_6(x) &= 32x^6 - 48x^4 + 18x^2 - 1\\
T_7(x) &= 64x^7 - 112x^5 + 56x^3 - 7x\\
T_8(x) &= 128x^8 - 256x^6 + 160x^4 - 32x^2 + 1\\
T_9(x) &= 256x^9 - 576x^7 + 432x^5 - 120x^3 + 9x\\
T_{10}(x) &= 512x^{10} - 1280x^8 + 1120x^6 - 400x^4 + 50x^2 - 1\\
T_{11}(x) &= 1024x^{11} - 2816x^9 + 2816x^7 - 1232x^5 + 220x^3 - 11x
\end{aligned}$$

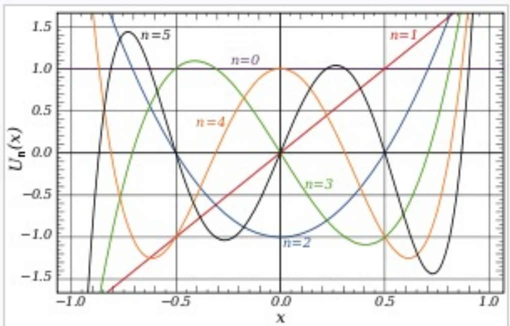


The first few Chebyshev polynomials of the first kind in the domain $-1 < x < 1$: The flat T_0, T_1, T_2, T_3, T_4 and T_5.

Second kind [edit]

The first few Chebyshev polynomials of the second kind are OEIS: A053117

$$\begin{aligned}
U_0(x) &= 1\\
U_1(x) &= 2x\\
U_2(x) &= 4x^2 - 1\\
U_3(x) &= 8x^3 - 4x\\
U_4(x) &= 16x^4 - 12x^2 + 1\\
U_5(x) &= 32x^5 - 32x^3 + 6x\\
U_6(x) &= 64x^6 - 80x^4 + 24x^2 - 1\\
U_7(x) &= 128x^7 - 192x^5 + 80x^3 - 8x\\
U_8(x) &= 256x^8 - 448x^6 + 240x^4 - 40x^2 + 1\\
U_9(x) &= 512x^9 - 1024x^7 + 672x^5 - 160x^3 + 10x
\end{aligned}$$



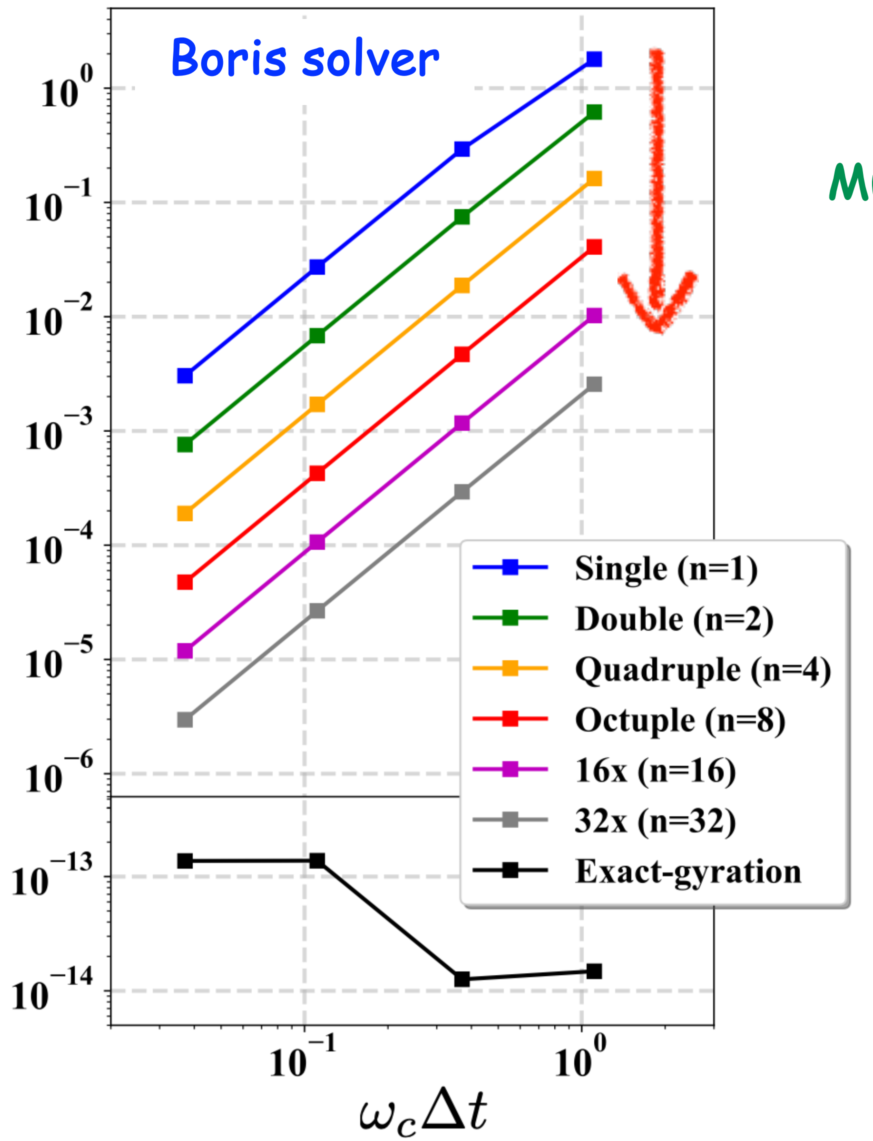
The first few Chebyshev polynomials of the second kind in the domain $-1 < x < 1$: The flat U_0, U_1, U_2, U_3, U_4 and U_5. Although not visible in the image, $U_n(1) = n + 1$ and $U_n(-1) = (n + 1)(-1)^n$.

https://en.wikipedia.org/wiki/Chebyshev_polynomials#Examples

Numerical test

Numerical error

$$\frac{\|\delta \boldsymbol{v}\|}{\|\boldsymbol{v}\|}$$



Multicycle Boris solver

$$\propto \left(\frac{\Delta t}{n}\right)^2$$

Zenitani & Kato 2020

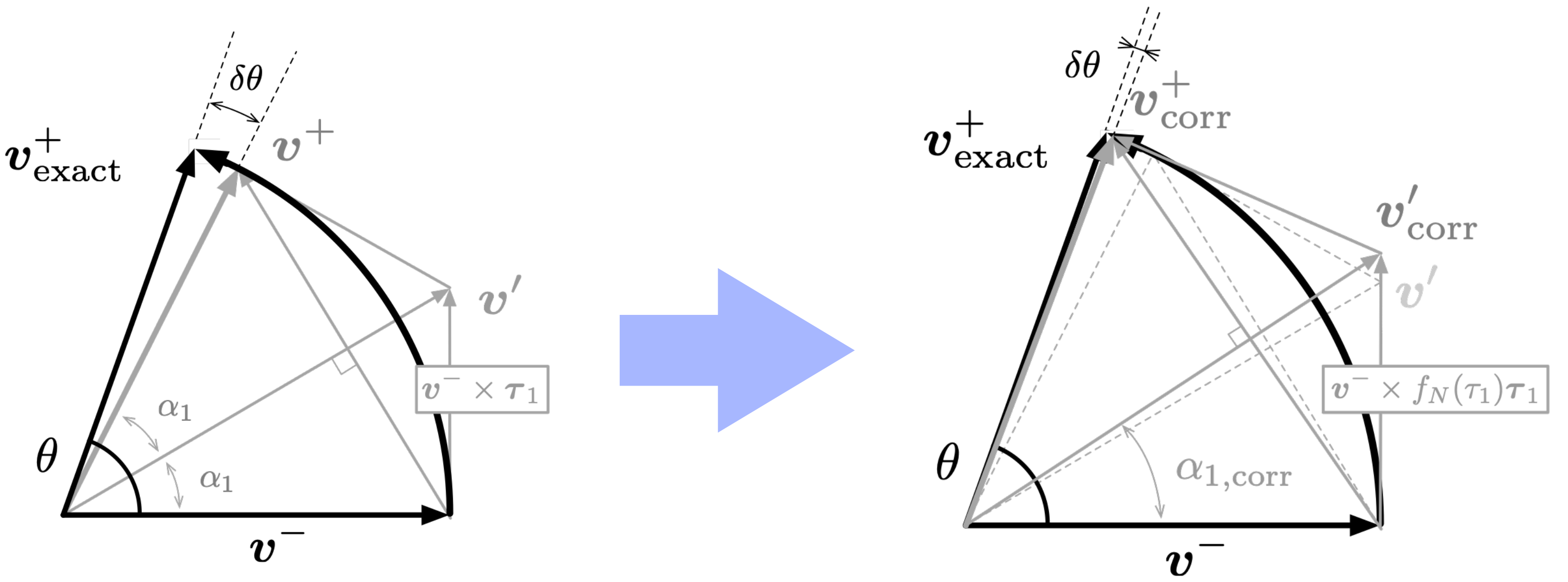
Solution 2 — Higher-order correction

- Boris (1970) has proposed a correction to the gyration part

element B vector $$\boldsymbol{t}_1 \equiv f_1 \frac{q\Delta t}{2m}\boldsymbol{B}$$

gyrophase correction $$\boldsymbol{t}_1 \leftarrow f_1 \boldsymbol{t}_1$$

$$f_1 \equiv \frac{\tan t_1}{t_1} = 1 + \frac{1}{3}t_1^2 + \frac{2}{15}t_1^4 + \frac{17}{315}t_1^6 + \frac{62}{2835}t_1^8 + \cdots$$



Anisotropic correction to E & B field

- Buneman-Boris method

approx. angle

$$\left(\boldsymbol{v}^{(1)} - \frac{\boldsymbol{E}\times\boldsymbol{B}}{B^2}\right) = \left(\boldsymbol{v}^{(0)} - \frac{\boldsymbol{E}\times\boldsymbol{B}}{B^2}\right)\cos\theta_1 + \left(\left(\boldsymbol{v}^{(0)} - \frac{\boldsymbol{E}\times\boldsymbol{B}}{B^2}\right)\times\hat{\boldsymbol{b}}\right)\sin\theta_1 + \left(\boldsymbol{v}^{(0)} - \frac{\boldsymbol{E}\times\boldsymbol{B}}{B^2}\right)_{\parallel}(1-\cos\theta_1) + \frac{q\boldsymbol{E}_{\parallel}}{m}\Delta t$$

Buneman 1967

- After gyrophase correction

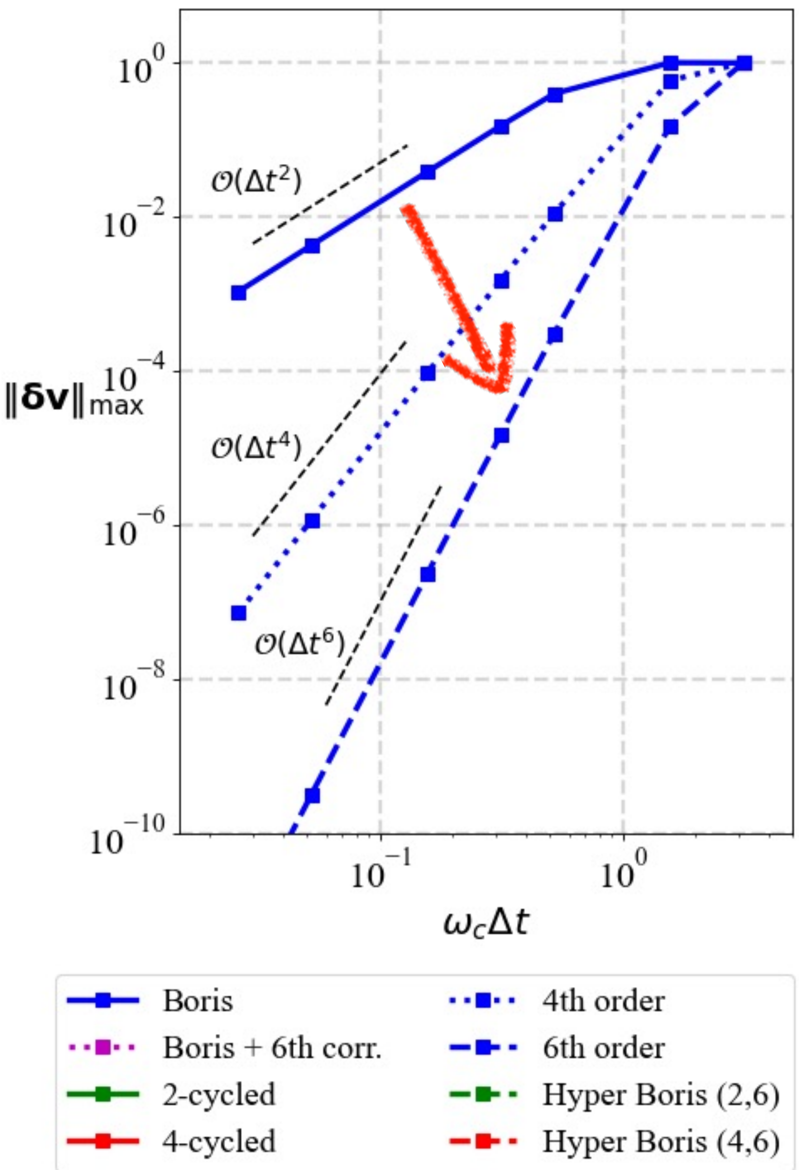
corrected angle

$$\left(\boldsymbol{v}^{(1)} - \frac{\boldsymbol{E}\times\boldsymbol{B}}{f_1 B^2}\right) = \left(\boldsymbol{v}^{(0)} - \frac{\boldsymbol{E}\times\boldsymbol{B}}{f_1 B^2}\right)\cos\theta_1^{\mathrm{corr}} + \left(\left(\boldsymbol{v}^{(0)} - \frac{\boldsymbol{E}\times\boldsymbol{B}}{f_1 B^2}\right)\times\hat{\boldsymbol{b}}\right)\sin\theta_1^{\mathrm{corr}} + \left(\boldsymbol{v}^{(0)} - \frac{\boldsymbol{E}\times\boldsymbol{B}}{f_1 B^2}\right)_{\parallel}(1-\cos\theta_1^{\mathrm{corr}}) + \frac{q\boldsymbol{E}_{\parallel}}{m}\Delta t$$

New 2nd-order error

One can further eliminate the new ExB error by amplifying $E_{\perp}$ AND by keeping $E_{//}$

Numerical test: higher-order correction



- In addition to the correction to B, we have applied the anisotropic correction to E

- We obtain higher-order results

$$\propto (\Delta t)^N$$

Solution 3 — Hybrid method

- 0. Buneman-Boris solver $\propto (\Delta t)^2$

- 1. Subcycling
  - It virtually repeats the 4-step procedure

$\propto \left(\frac{\Delta t}{n}\right)^2$

- 2. Higher-order correction
  - Modification before the 4-step procedure

$\propto (\Delta t)^N$

- 3. Hyper Boris solver
  - n-times cycling & Nth-order correction

Sample Recipe: (4_{cycle},$6_{\text{th-order}}$) solver

- 1. Define the element vectors

$$\boldsymbol{\tau}_4 \equiv \frac{q\Delta t}{8m}\boldsymbol{B} \qquad \boldsymbol{\varepsilon}_4 \equiv \frac{q\Delta t}{8m}\boldsymbol{E}$$

- 2. Higher-order correction

$$\boldsymbol{t}_4 \equiv \left(1+\frac{1}{3}\tau_4^2+\frac{2}{15}\tau_4^4\right)\boldsymbol{\tau}_4 \qquad \boldsymbol{e}_4 \equiv \left(1+\frac{1}{3}\tau_4^2+\frac{2}{15}\tau_4^4\right)\boldsymbol{\varepsilon}_4 - \left(\frac{1}{3}+\frac{2}{15}\tau_4^2\right)(\boldsymbol{\varepsilon}_4\cdot\boldsymbol{\tau}_4)\boldsymbol{\tau}_4$$

- 3. Calculate the coefficients

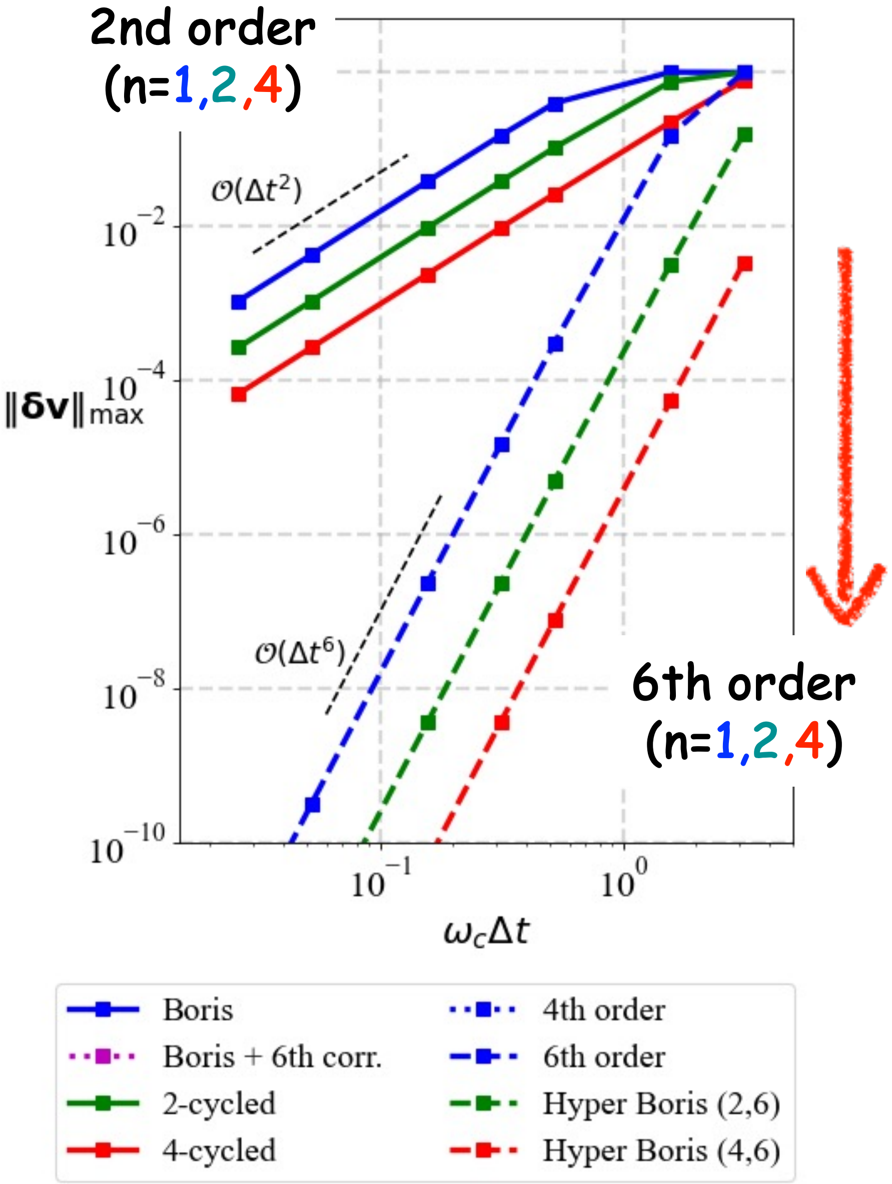
$$c_{41}=\frac{1-28t_4^2+70t_4^4-28t_4^6+t_4^8}{(1+t_4^2)^4}, \quad c_{42}=\frac{8(1-7t_4^2+7t_4^4-t^6)}{(1+t_4^2)^4}, \quad c_{43}=\frac{32(1-t_4^2)^2}{(1+t_4^2)^4},$$

$$c_{44}=c_{42}, \qquad c_{45}=c_{43}, \qquad c_{46}=\frac{8(11-t_4^2+5t_4^4+t_4^6)}{(1+t_4^2)^4}$$

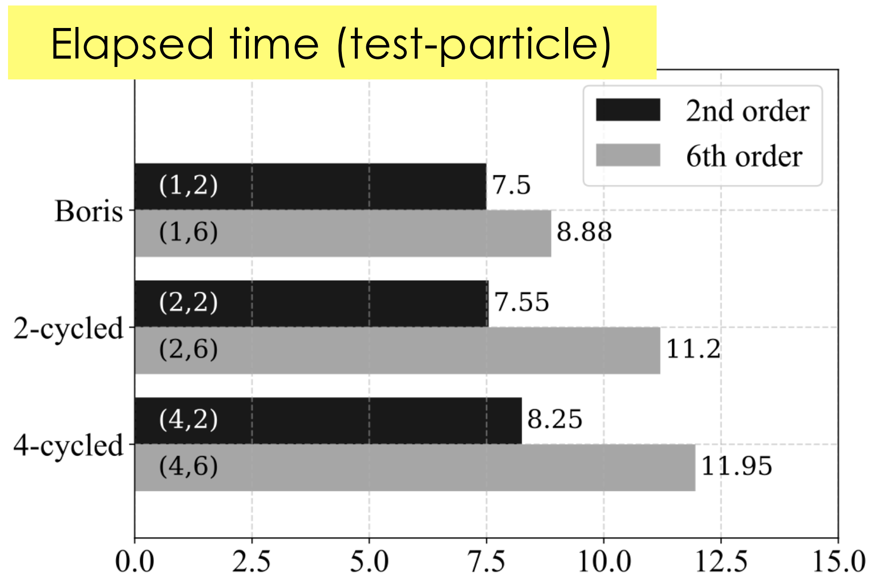
- 4. Calculate the velocity at the next timestep

$$\boldsymbol{v}^{t+\Delta t}=c_{41}\boldsymbol{v}^t+c_{42}\left(\boldsymbol{v}^t\times\boldsymbol{t}_4\right)+c_{43}(\boldsymbol{v}^t\cdot\boldsymbol{t}_4)\boldsymbol{t}_4$$
$$+c_{44}\boldsymbol{e}_4+c_{45}\left(\boldsymbol{e}_4\times\boldsymbol{t}_4\right)+c_{46}(\boldsymbol{e}_4\cdot\boldsymbol{t}_4)\boldsymbol{t}_4$$

Hyper Boris solver: Numerical tests (1)

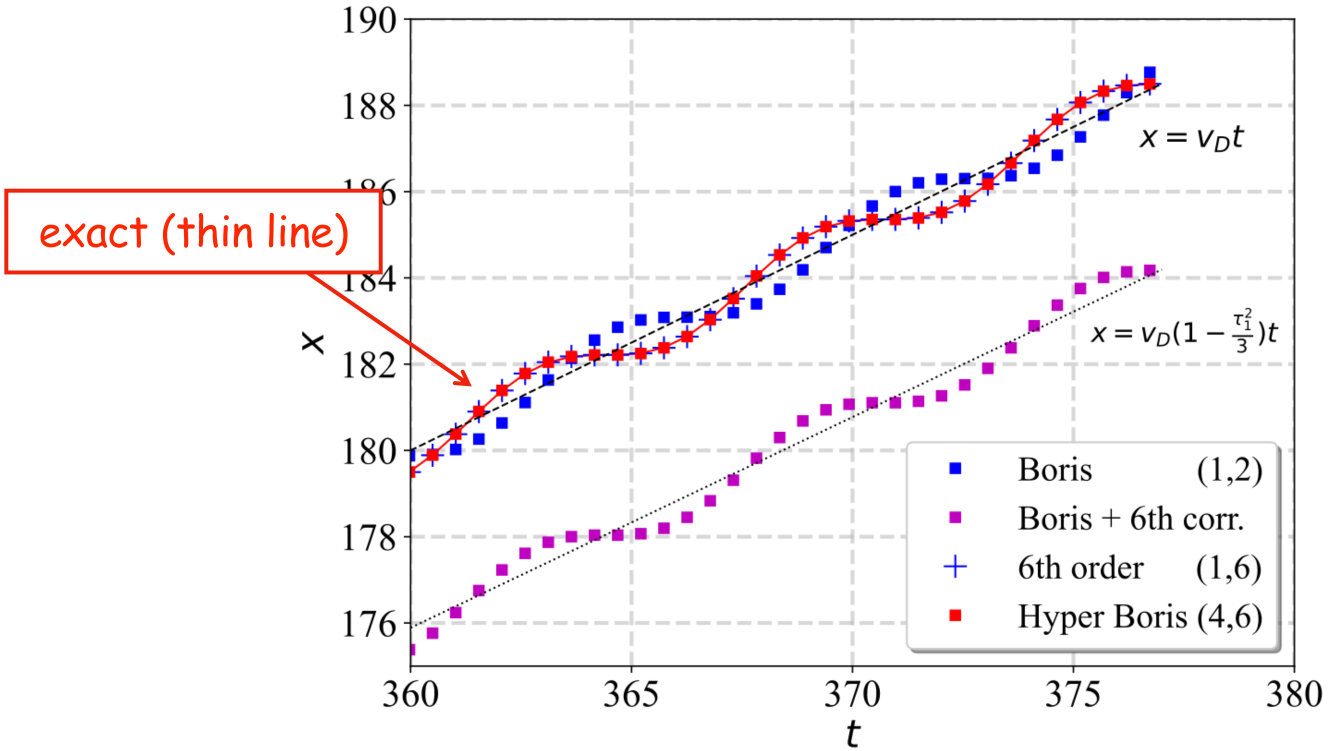


- Numerical accuracy of $\propto \left(\frac{\Delta t}{n}\right)^N$
- Affordable computational costs



Hyper Boris solver: Numerical tests (2)

- Long-term ExB drift
  - Popular gryophase correction slows down the drift speed
  - Our correction preserves the drift speed



Summary

- 0. Buneman-Boris solver $\propto (\Delta t)^2$

- 1. Subcycling
  - Multicycle formula with Chebyshev polynomials for arbitrary n $\propto \left(\frac{\Delta t}{n}\right)^2$

- 2. Higher-order correction
  - Anisotropic correction to the electric field $\propto (\Delta t)^N$

- 3. Hyper Boris solver $\propto \left(\frac{\Delta t}{n}\right)^N$
  - n-times cycling & Nth-order cor.
  - Applicable to any problem of $\frac{d\boldsymbol{v}}{dt} = \boldsymbol{F} + \boldsymbol{v} \times \boldsymbol{R}$

Better simulations for kinetic plasma problems

Frequently Asked Questions

- 1. Is the hyper Boris solver symplectic?
  - No. Boris-type solvers are not symplectic.
- 2. Does it have a long-term stability?
  - Yes. Because it preserves the phase-space volume.
- 3. What hyperparameters do you recommend?
  - I recommend (2-cycle, 6th-order) or (4-cycle, 6th) solvers.
- 4. Can the hyper Boris solver solve relativistic motion?
  - No. We can actually use multicycling, but we can no longer use the multicycle formula.
- 5. What should we use in the relativistic regime?
  - For a moment, I recommend the Vay solver (Vay 2008) or the quadruple Boris solver (Zenitani & Kato 2020).
  - We need to develop new solvers.

Coefficients (numerators only)

```
c11:  1 - t**2
c12:  2
c13:  2
c14:  2
c15:  2
c16:  2

c21:  t**4 - 6*t**2 + 1
c22:  4 - 4*t**2
c23:  8
c24:  4 - 4*t**2
c25:  8
c26:  4*t**2 + 12

c31:  -t**6 + 15*t**4 - 15*t**2 + 1
c32:  6*t**4 - 20*t**2 + 6
c33:  -2*t**4 + 4*t**2 + 6
c34:  6*t**4 - 20*t**2 + 6
c35:  -2*t**4 + 4*t**2 + 6
c36:  6*t**4 + 12*t**2 + 38

c41:  t**8 - 28*t**6 + 70*t**4 - 28*t**2 + 1
c42:  -8*t**6 + 56*t**4 - 56*t**2 + 8
c43:  32*t**4 - 64*t**2 + 32
c44:  -8*t**6 + 56*t**4 - 56*t**2 + 8
c45:  32*t**4 - 64*t**2 + 32
c46:  8*t**6 + 40*t**4 - 8*t**2 + 88

c51:  -t**10 + 45*t**8 - 210*t**6 + 210*t**4 - 45*t**2 + 1
c52:  10*t**8 - 120*t**6 + 252*t**4 - 120*t**2 + 10
c53:  2*t**8 - 16*t**6 - 28*t**4 + 10
c54:  10*t**8 - 120*t**6 + 252*t**4 - 120*t**2 + 10
c55:  2*t**8 - 16*t**6 - 28*t**4 + 10
c56:  10*t**8 + 40*t**6 + 220*t**4 - 152*t**2 + 170

c61:  t**12 - 66*t**10 + 495*t**8 - 924*t**6 + 495*t**4 - 66*t**2 + 1
c62:  -12*t**10 + 220*t**8 - 792*t**6 + 792*t**4 - 220*t**2 + 12
c63:  72*t**8 - 480*t**6 + 944*t**4 - 480*t**2 + 72
c64:  -12*t**10 + 220*t**8 - 792*t**6 + 792*t**4 - 220*t**2 + 12
c65:  72*t**8 - 480*t**6 + 944*t**4 - 480*t**2 + 72
```

Coefficients (numerators only)

```
c71:  -t**14 + 91*t**12 - 1001*t**10 + 3003*t**8 - 3003*t**6 + 1001*t**4 - 91*t**2 + 1
c72:  14*t**12 - 364*t**10 + 2002*t**8 - 3432*t**6 + 2002*t**4 - 364*t**2 + 14
c73:  -2*t**12 + 36*t**10 + 50*t**8 - 72*t**6 - 126*t**4 - 28*t**2 + 14
c74:  14*t**12 - 364*t**10 + 2002*t**8 - 3432*t**6 + 2002*t**4 - 364*t**2 + 14
c75:  -2*t**12 + 36*t**10 + 50*t**8 - 72*t**6 - 126*t**4 - 28*t**2 + 14
c76:  14*t**12 + 84*t**10 + 658*t**8 - 1512*t**6 + 3922*t**4 - 1708*t**2 + 462

c81:  t**16 - 120*t**14 + 1820*t**12 - 8008*t**10 + 12870*t**8 - 8008*t**6 + 1820*t**4 - 120*t**2 + 1
c82:  -16*t**14 + 560*t**12 - 4368*t**10 + 11440*t**8 - 11440*t**6 + 4368*t**4 - 560*t**2 + 16
c83:  128*t**12 - 1792*t**10 + 8064*t**8 - 12800*t**6 + 8064*t**4 - 1792*t**2 + 128
c84:  -16*t**14 + 560*t**12 - 4368*t**10 + 11440*t**8 - 11440*t**6 + 4368*t**4 - 560*t**2 + 16
c85:  128*t**12 - 1792*t**10 + 8064*t**8 - 12800*t**6 + 8064*t**4 - 1792*t**2 + 128
c86:  16*t**14 + 144*t**12 - 112*t**10 + 5264*t**8 - 10320*t**6 + 12336*t**4 - 3920*t**2 + 688

c91:  -t**18 + 153*t**16 - 3060*t**14 + 18564*t**12 - 43758*t**10 + 43758*t**8 - 18564*t**6 + 3060*t**4 - 153*t**2 + 1
c92:  18*t**16 - 816*t**14 + 8568*t**12 - 31824*t**10 + 48620*t**8 - 31824*t**6 + 8568*t**4 - 816*t**2 + 18
c93:  2*t**16 - 64*t**14 - 24*t**12 + 416*t**10 + 572*t**8 - 312*t**4 - 96*t**2 + 18
c94:  18*t**16 - 816*t**14 + 8568*t**12 - 31824*t**10 + 48620*t**8 - 31824*t**6 + 8568*t**4 - 816*t**2 + 18
c95:  2*t**16 - 64*t**14 - 24*t**12 + 416*t**10 + 572*t**8 - 312*t**4 - 96*t**2 + 18
c96:  18*t**16 + 144*t**14 + 1464*t**12 - 7056*t**10 + 34092*t**8 - 46352*t**6 + 33336*t**4 - 7920*t**2 + 978

c101:  t**20 - 190*t**18 + 4845*t**16 - 38760*t**14 + 125970*t**12 - 184756*t**10 + 125970*t**8 - 38760*t**6 + 4845*t**4 - 190*t**2 + 1
c102:  -20*t**18 + 1140*t**16 - 15504*t**14 + 77520*t**12 - 167960*t**10 + 167960*t**8 - 77520*t**6 + 15504*t**4 - 1140*t**2 + 20
c103:  200*t**16 - 4800*t**14 + 38880*t**12 - 125760*t**10 + 185008*t**8 - 125760*t**6 + 38880*t**4 - 4800*t**2 + 200
c104:  -20*t**18 + 1140*t**16 - 15504*t**14 + 77520*t**12 - 167960*t**10 + 167960*t**8 - 77520*t**6 + 15504*t**4 - 1140*t**2 + 20
c105:  200*t**16 - 4800*t**14 + 38880*t**12 - 125760*t**10 + 185008*t**8 - 125760*t**6 + 38880*t**4 - 4800*t**2 + 200
c106:  20*t**18 + 220*t**16 - 240*t**14 + 17904*t**12 - 73320*t**10 + 173000*t**8 - 163760*t**6 + 79920*t**4 - 14604*t**2 + 1340

c111:  -t**22 + 231*t**20 - 7315*t**18 + 74613*t**16 - 319770*t**14 + 646646*t**12 - 646646*t**10 + 319770*t**8 - 74613*t**6 + 7315*t**4 - 231*t**2 + 1
c112:  22*t**20 - 1540*t**18 + 26334*t**16 - 170544*t**14 + 497420*t**12 - 705432*t**10 + 497420*t**8 - 170544*t**6 + 26334*t**4 - 1540*t**2 + 22
c113:  -2*t**20 + 100*t**18 - 130*t**16 - 1296*t**14 - 1220*t**12 + 1560*t**10 + 2860*t**8 + 880*t**6 - 506*t**4 - 220*t**2 + 22
c114:  22*t**20 - 1540*t**18 + 26334*t**16 - 170544*t**14 + 497420*t**12 - 705432*t**10 + 497420*t**8 - 170544*t**6 + 26334*t**4 - 1540*t**2 + 22
c115:  -2*t**20 + 100*t**18 - 130*t**16 - 1296*t**14 - 1220*t**12 + 1560*t**10 + 2860*t**8 + 880*t**6 - 506*t**4 - 220*t**2 + 22
c116:  22*t**20 + 220*t**18 + 2750*t**16 - 22704*t**14 + 177804*t**12 - 487256*t**10 + 715596*t**8 - 490160*t**6 + 174174*t**4 - 25124*t**2 + 1782

c121:  t**24 - 276*t**22 + 10626*t**20 - 134596*t**18 + 735471*t**16 - 1961256*t**14 + 2704156*t**12 - 1961256*t**10 + 735471*t**8 - 134596*t**6 + 10626*t**4 - 276*t**2 + 1
c122:  -24*t**22 + 2024*t**20 - 42504*t**18 + 346104*t**16 - 1307504*t**14 + 2496144*t**12 - 2496144*t**10 + 1307504*t**8 - 346104*t**6 + 42504*t**4 - 2024*t**2 + 24
c123:  288*t**20 - 10560*t**18 + 134816*t**16 - 734976*t**14 + 1962048*t**12 - 2703232*t**10 + 1962048*t**8 - 734976*t**6 + 134816*t**4 - 10560*t**2 + 288
c124:  -24*t**22 + 2024*t**20 - 42504*t**18 + 346104*t**16 - 1307504*t**14 + 2496144*t**12 - 2496144*t**10 + 1307504*t**8 - 346104*t**6 + 42504*t**4 - 2024*t**2 + 24
c125:  288*t**20 - 10560*t**18 + 134816*t**16 - 734976*t**14 + 1962048*t**12 - 2703232*t**10 + 1962048*t**8 - 734976*t**6 + 134816*t**4 - 10560*t**2 + 288
c126:  24*t**22 + 312*t**20 - 440*t**18 + 47784*t**16 - 334224*t**14 + 1326512*t**12 - 2473968*t**10 + 2515152*t**8 - 1295624*t**6 + 351384*t**4 - 40920*t**2 + 2312

c131:  -t**26 + 325*t**24 - 14950*t**22 + 230230*t**20 - 1562275*t**18 + 5311735*t**16 - 9657700*t**14 + 9657700*t**12 - 5311735*t**10 + 1562275*t**8 - 230230*t**6 + 14950*t**4 - 325*t**2 + 1
c132:  26*t**24 - 2600*t**22 + 65780*t**20 - 657800*t**18 + 3124550*t**16 - 7726160*t**14 + 10400600*t**12 - 7726160*t**10 + 3124550*t**8 - 657800*t**6 + 65780*t**4 - 2600*t**2 + 26
c133:  2*t**24 - 144*t**22 + 524*t**20 + 2848*t**18 + 342*t**16 - 10336*t**14 - 12920*t**12 + 8398*t**8 + 3952*t**6 - 468*t**4 - 416*t**2 + 26
c134:  26*t**24 - 2600*t**22 + 65780*t**20 - 657800*t**18 + 3124550*t**16 - 7726160*t**14 + 10400600*t**12 - 7726160*t**10 + 3124550*t**8 - 657800*t**6 + 65780*t**4 - 2600*t**2 + 26
c135:  2*t**24 - 144*t**22 + 524*t**20 + 2848*t**18 + 342*t**16 - 10336*t**14 - 12920*t**12 + 8398*t**8 + 3952*t**6 - 468*t**4 - 416*t**2 + 26
c136:  26*t**24 + 312*t**22 + 4628*t**20 - 58344*t**18 + 676390*t**16 - 3091088*t**14 + 7770776*t**12 - 10355984*t**10 + 7759622*t**8 - 3105960*t**6 + 665236*t**4 - 63752*t**2 + 2938

c141:  t**28 - 378*t**26 + 20475*t**24 - 376740*t**22 + 3108105*t**20 - 13123110*t**18 + 30421755*t**16 - 40116600*t**14 + 30421755*t**12 - 13123110*t**10 + 3108105*t**8 - 376740*t**6 + 20475*t**4 - 378*t**2 + 1
c142:  -28*t**26 + 3276*t**24 - 98280*t**22 + 1184040*t**20 - 6906900*t**18 + 21474180*t**16 - 37442160*t**14 + 37442160*t**12 - 21474180*t**10 + 6906900*t**8 - 1184040*t**6 + 98280*t**4 - 3276*t**2 + 28
c143:  392*t**24 - 20384*t**22 + 377104*t**20 - 3107104*t**18 + 13125112*t**16 - 30418752*t**14 + 40120032*t**12 - 30418752*t**10 + 13125112*t**8 - 3107104*t**6 + 377104*t**4 - 20384*t**2 + 392
c144:  -28*t**26 + 3276*t**24 - 98280*t**22 + 1184040*t**20 - 6906900*t**18 + 21474180*t**16 - 37442160*t**14 + 37442160*t**12 - 21474180*t**10 + 6906900*t**8 - 1184040*t**6 + 98280*t**4 - 3276*t**2 + 28
c145:  392*t**24 - 20384*t**22 + 377104*t**20 - 3107104*t**18 + 13125112*t**16 - 30418752*t**14 + 40120032*t**12 - 30418752*t**10 + 13125112*t**8 - 3107104*t**6 + 377104*t**4 - 20384*t**2 + 392
c146:  28*t**26 + 420*t**24 - 728*t**22 + 108472*t**20 - 1156012*t**18 + 6962956*t**16 - 21390096*t**14 + 37538256*t**12 - 37358076*t**10 + 21530236*t**8 - 6878872*t**6 + 1194232*t**4 - 95732*t**2 + 3668

c151:  -t**30 + 435*t**28 - 27405*t**26 + 593775*t**24 - 5852925*t**22 + 30045015*t**20 - 86493225*t**18 + 145422675*t**16 - 145422675*t**14 + 86493225*t**12 - 30045015*t**10 + 5852925*t**8 - 593775*t**6 +
27405*t**4 - 435*t**2 + 1
c152:  30*t**28 - 4060*t**26 + 142506*t**24 - 2035800*t**22 + 14307150*t**20 - 54627300*t**18 + 119759850*t**16 - 155117520*t**14 + 119759850*t**12 - 54627300*t**10 + 14307150*t**8 - 2035800*t**6 + 142506*t**4 -
4060*t**2 + 30
c153:  -2*t**28 + 196*t**26 - 1302*t**24 - 4760*t**22 + 7150*t**20 + 37884*t**18 + 30058*t**16 - 36176*t**14 - 67830*t**12 - 23940*t**10 + 15246*t**8 + 11112*t**6 + 266*t**4 - 700*t**2 + 30
c154:  30*t**28 - 4060*t**26 + 142506*t**24 - 2035800*t**22 + 14307150*t**20 - 54627300*t**18 + 119759850*t**16 - 155117520*t**14 + 119759850*t**12 - 54627300*t**10 + 14307150*t**8 - 2035800*t**6 + 142506*t**4 -
4060*t**2 + 30
c155:  -2*t**28 + 196*t**26 - 1302*t**24 - 4760*t**22 + 7150*t**20 + 37884*t**18 + 30058*t**16 - 36176*t**14 - 67830*t**12 - 23940*t**10 + 15246*t**8 + 11112*t**6 + 266*t**4 - 700*t**2 + 30
c156:  30*t**28 + 420*t**26 + 7210*t**24 - 128856*t**22 + 2076750*t**20 - 14217060*t**18 + 54777450*t**16 - 119566800*t**14 + 155310570*t**12 - 119609700*t**10 + 54717390*t**8 - 14266200*t**6 + 2049450*t**4 -
139356*t**2 + 4510

c161:  t**32 - 496*t**30 + 35960*t**28 - 906192*t**26 + 10518300*t**24 - 64512240*t**22 + 225792840*t**20 - 471435600*t**18 + 601080390*t**16 - 471435600*t**14 + 225792840*t**12 - 64512240*t**10 + 10518300*t**8 -
906192*t**6 + 35960*t**4 - 496*t**2 + 1
c162:  -32*t**30 + 4960*t**28 - 201376*t**26 + 3365856*t**24 - 28048800*t**22 + 129024480*t**20 - 347373600*t**18 + 565722720*t**16 - 565722720*t**14 + 347373600*t**12 - 129024480*t**10 + 28048800*t**8 -
3365856*t**6 + 201376*t**4 - 4960*t**2 + 32
c163:  512*t**28 - 35840*t**26 + 906752*t**24 - 10516480*t**22 + 64516608*t**20 - 225784832*t**18 + 471447040*t**16 - 601067520*t**14 + 471447040*t**12 - 225784832*t**10 + 64516608*t**8 - 10516480*t**6 +
906752*t**4 - 35840*t**2 + 512
c164:  -32*t**30 + 4960*t**28 - 201376*t**26 + 3365856*t**24 - 28048800*t**22 + 129024480*t**20 - 347373600*t**18 + 565722720*t**16 - 565722720*t**14 + 347373600*t**12 - 129024480*t**10 + 28048800*t**8 -
3365856*t**6 + 201376*t**4 - 4960*t**2 + 32
c165:  512*t**28 - 35840*t**26 + 906752*t**24 - 10516480*t**22 + 64516608*t**20 - 225784832*t**18 + 471447040*t**16 - 601067520*t**14 + 471447040*t**12 - 225784832*t**10 + 64516608*t**8 - 10516480*t**6 +
906752*t**4 - 35840*t**2 + 512
c166:  32*t**30 + 544*t**28 - 1120*t**26 + 219296*t**24 - 3307616*t**22 + 28188576*t**20 - 128768224*t**18 + 347739680*t**16 - 565310880*t**14 + 566088800*t**12 - 347117344*t**10 + 129164256*t**8 - 27990560*t**6
+ 3383776*t**4 - 197536*t**2 + 5472
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