Loading relativistic kappa distributions in particle-in-cell (PIC) simulations

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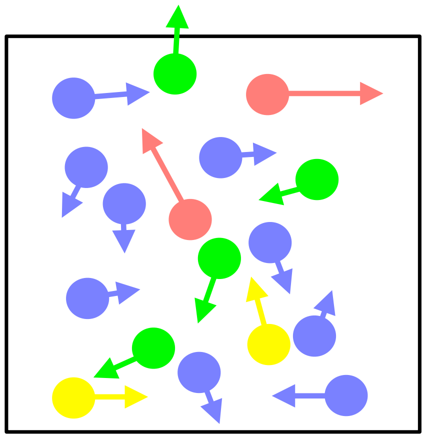
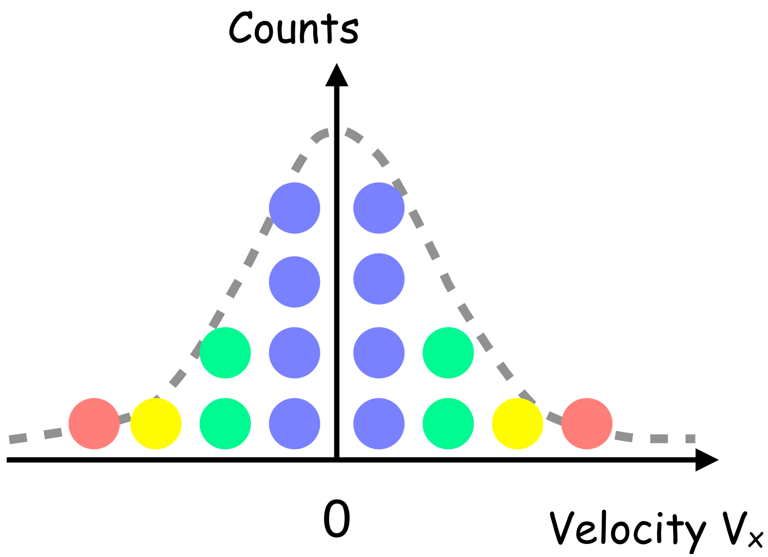




Kappa distributions

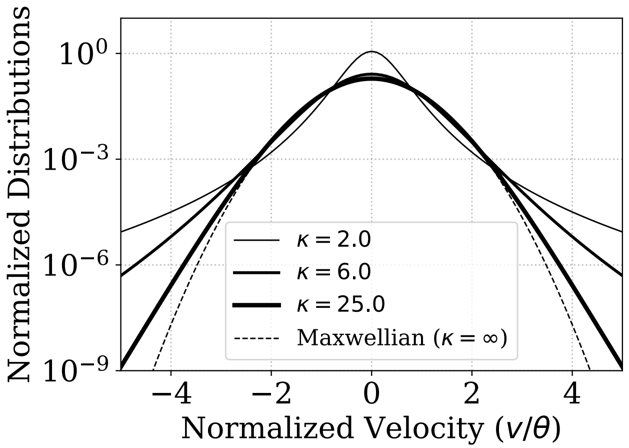
- Kappa distribution

$$f(\boldsymbol{v})d^3v = \frac{N}{(\pi\kappa\theta^2)^{3/2}}\frac{\Gamma(\kappa+1)}{\Gamma(\kappa-1/2)}\left(1+\frac{v^2}{\kappa\theta^2}\right)^{-(\kappa+1)}d^3v$$

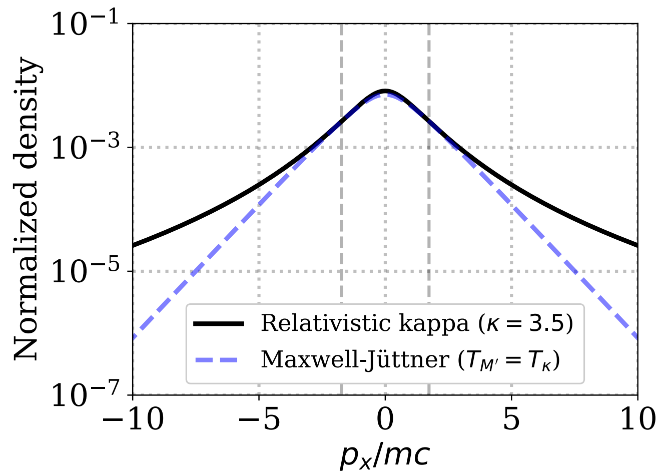
- Relativistic Kappa distribution

$$f_{\mathrm{RK}}(\boldsymbol{p})d^3p \equiv A\left(1+\frac{(\gamma-1)mc^2}{\kappa T_\kappa}\right)^{-(\kappa+1)}d^3p$$

$$\boldsymbol{p} = m\gamma\boldsymbol{v} \quad \gamma = [1-(v/c)^2]^{-1/2}$$



relativistic



How to generate a relativistic kappa distribution in PIC simulations?

Some mathematics

- Relativistic Kappa distribution

$$f_{\rm RK}(p)dp = A(\kappa,t)\left(1+\frac{\gamma-1}{\kappa t}\right)^{-(\kappa+1)} 4\pi p^2\, dp$$

$$x \equiv \frac{\mathcal{E}_{\rm kin}}{mc^2} = \gamma - 1$$

Normalization factors

$$A(\kappa,T_\kappa) = \frac{N_\kappa \Gamma\left(\kappa+\frac{1}{2}\right)}{(2\pi m\kappa T_\kappa)^{3/2}(\kappa+1)\;\Gamma(\kappa-2)\;{}_2F_1\left(-\frac{3}{2},\frac{5}{2};\kappa+\frac{1}{2};1-\frac{\kappa T_\kappa}{2mc^2}\right)}$$

$$S(\kappa,t) \equiv \frac{\sqrt{2\pi}}{2}\Gamma\left(\kappa-\frac{1}{2}\right) + a\sqrt{\kappa t}\;\Gamma(\kappa-1) + b\,\frac{3\sqrt{2\pi}}{4}(\kappa t)\;\left(\kappa-\frac{3}{2}\right) + 2(\kappa t)^{3/2}\;\Gamma(\kappa-2)$$

- Our expression

$$f_{\rm RK}(x)dx = \frac{4\pi A(\kappa,t)S(\kappa,t)}{\Gamma(\kappa+1)}(\kappa t)^{3/2}\left(\sum_{i=3}^{6}\pi_i(\kappa,t)\;\mathrm{B}'\left(x;\frac{i}{2},\kappa+1-\frac{i}{2},1,\kappa t\right)\right) R(x)\;dx$$

Probability

$$\sum_{i=3}^{6}\pi_i(\kappa,t) = 1$$

Generalized Beta prime distribution

$$\mathrm{B}'(x;\alpha,\beta,p,q) = \frac{p}{qB(\alpha,\beta)}\left(\frac{x}{q}\right)^{\alpha p-1}\left(1+\left(\frac{x}{q}\right)^{p}\right)^{-(\alpha+\beta)}$$

Rejection function

$$R(x;a,b) \equiv \frac{(1+x)\sqrt{x+2}}{\sqrt{2}+ax^{1/2}+b\sqrt{2}x+x^{3/2}}$$

The algorithm

Main procedure

$a \leftarrow 0.56, \quad b \leftarrow 0.35, \quad R_0 \leftarrow 0.95$

compute π_3, π_4, π_5 for given κ, t using Eqs. (40)–(42)

repeat

  generate $X_1, X_2 \sim U(0,1)$ — Probabilistic switch

  if $X_1 < \pi_3$ **then** $i \leftarrow 3$

  elseif $X_1 < \pi_3 + \pi_4$ **then** $i \leftarrow 4$

  elseif $X_1 < \pi_3 + \pi_4 + \pi_5$ **then** $i \leftarrow 5$

  else $i \leftarrow 6$

  endif

  generate $X_3 \sim \mathrm{Ga}(i/2, 1), X_4 \sim \mathrm{Ga}(\kappa + 1 - i/2, 1)$

  $x \leftarrow \kappa t \times \dfrac{X_3}{X_4}$ — Beta prime distribution

until $X_2 < R_0$ **or** $X_2 < R(x; a, b)$ — Rejection

generate $X_5, X_6 \sim U(0,1)$

$p \leftarrow \sqrt{x\,(x+2)}$

$p_x \leftarrow p\,(2X_5 - 1)$

$p_y \leftarrow 2p\sqrt{X_5(1 - X_5)}\cos(2\pi X_6)$

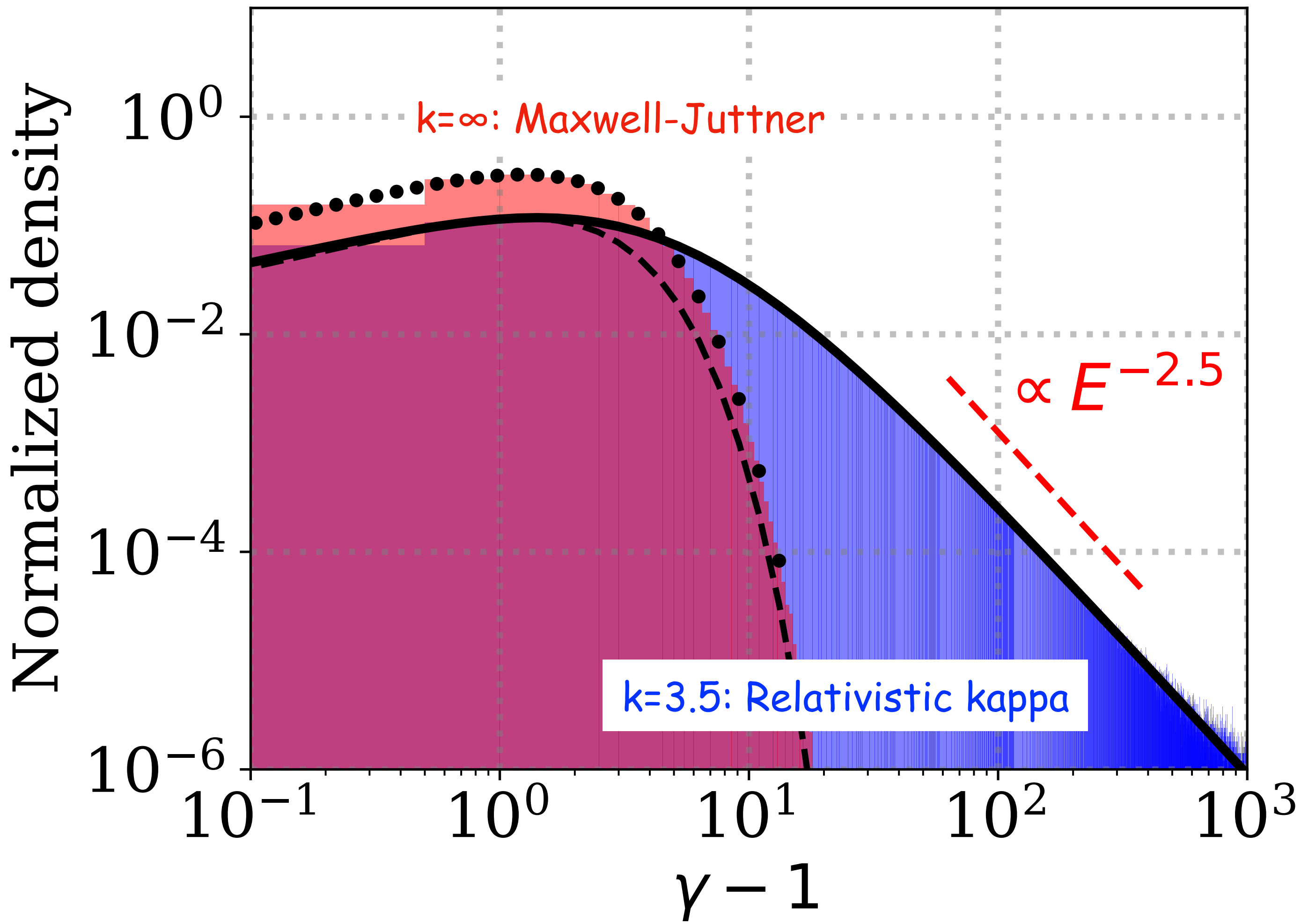
$p_z \leftarrow 2p\sqrt{X_5(1 - X_5)}\sin(2\pi X_6)$

- Beta prime distribution can be generated from 2 gamma distributions

$$X_{\mathrm{B}'(\alpha,\beta)} = \frac{X_{\Gamma(\alpha,\delta)}}{X_{\Gamma(\beta,\delta)}} = \frac{X_{\Gamma(\alpha,1)}}{X_{\Gamma(\beta,1)}}$$

- No need for the normalization factors
- Gamma variates can be easily generated

Numerical test

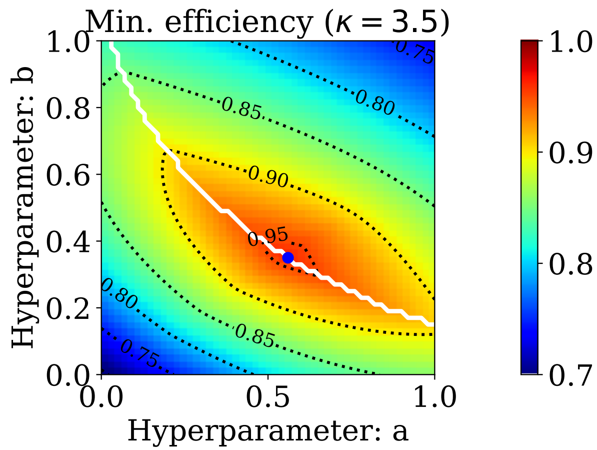


Zenitani & Nakano 2022

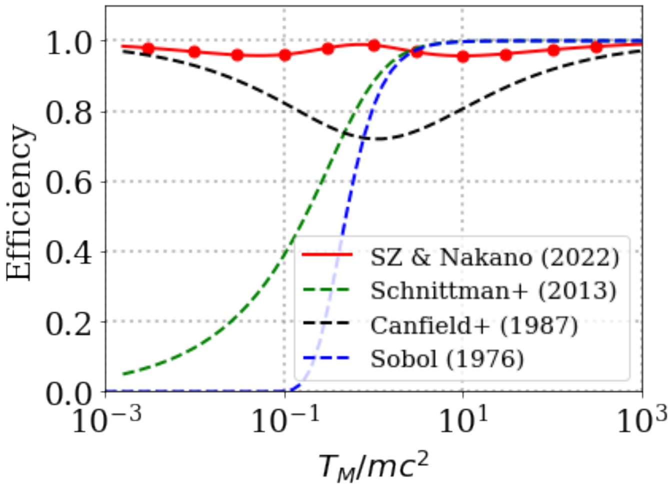
Improved rejection function

- Modified Canfield (1987) method
  - New hyperparameters

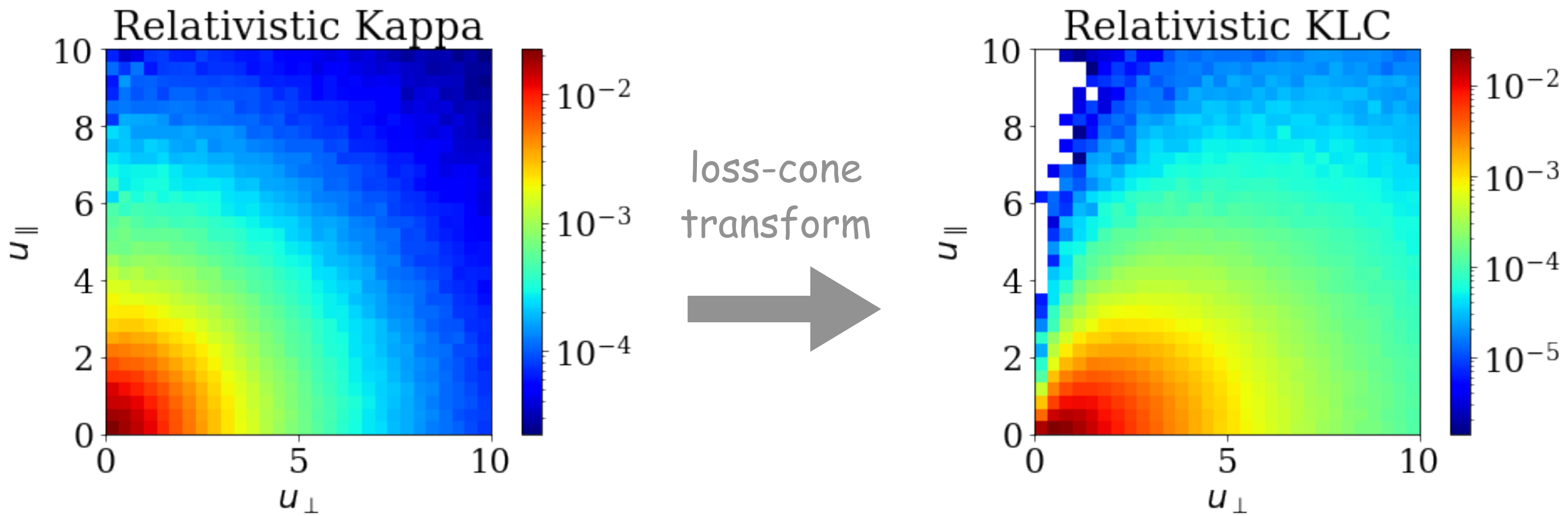
$$R(x; a, b) \equiv \frac{(1+x)\sqrt{x+2}}{\sqrt{2} + a x^{1/2} + b\sqrt{2}x + x^{3/2}}$$



- Acceptance efficiency (for MJ)



Relativistic kappa loss-cone (KLC)



- Loss-cone transform
  for <u>isotropic</u> distributions

$$f_{RKLC} = \frac{2}{\sqrt{\pi}} \frac{\Gamma(j+3/2)}{\Gamma(j+1)} \Big(\sin \alpha \Big)^{2j} f_{RK}$$

α: pitch angle

Summary

- 1. Beta prime distributions --> relativistic kappa distribution
- 2. Efficient rejection function: 70% --> 95%
- 3. Loss-cone transform --> relativistic kappa loss-cone dist.

- References:
  - Zenitani & Nakano, *Phys. Plasmas* **29**, 113904 (2022)
  - Zenitani & Nakano, in prep.

プラズマ粒子シミュレーションのための相対論的カッパ分布の乱数生成法

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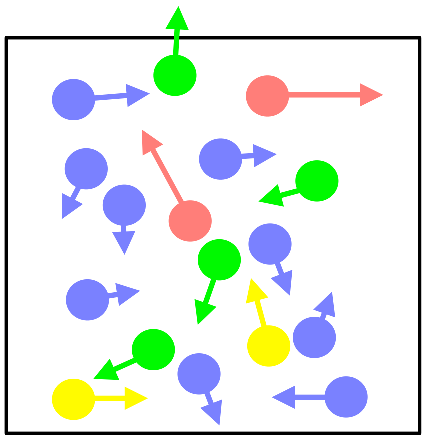
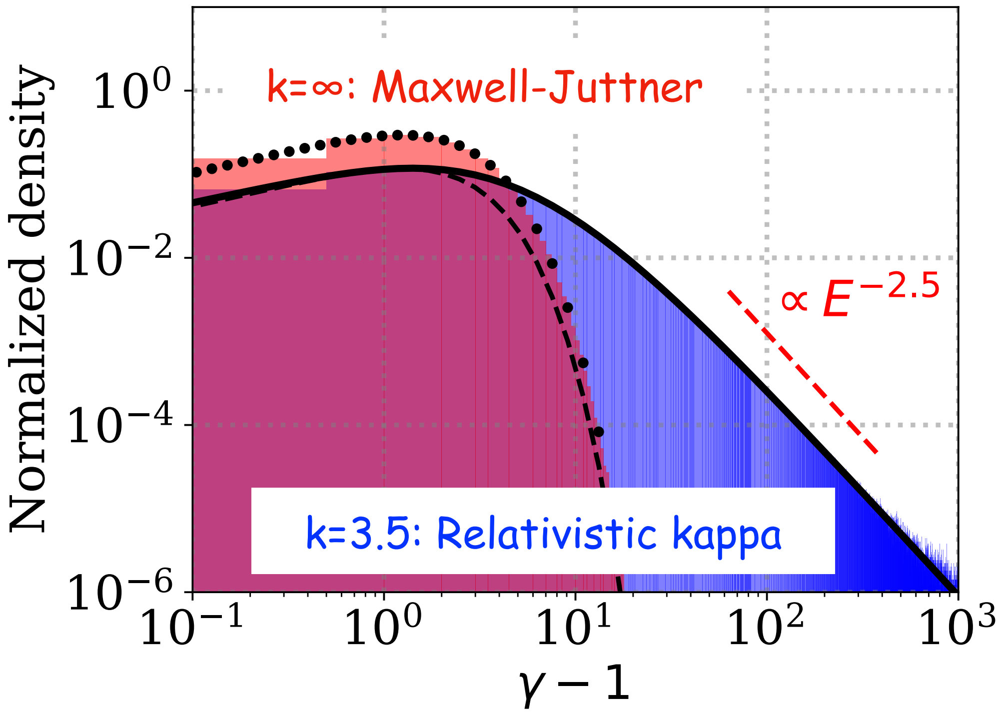
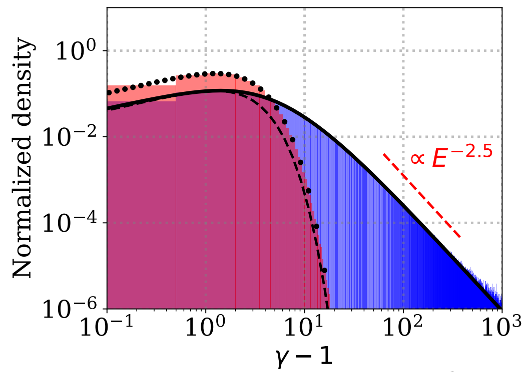




(1) ベータプライム分布を重ね合わせて
相対論的カッパ分布の乱数生成を実現

$$f_{\rm RK}(p)dp = A(\kappa,t)\left(1+\frac{\gamma-1}{\kappa t}\right)^{-(\kappa+1)} 4\pi p^2\ dp$$



$$\Rightarrow f_{\rm RK}(x)dx = \frac{4\pi A(\kappa,t)S(\kappa,t)}{\Gamma(\kappa+1)}(\kappa t)^{3/2}\left(\sum_{i=3}^{6}\pi_i(\kappa,t)\,{\rm B}'\left(x;\frac{i}{2},\kappa+1-\frac{i}{2},1,\kappa t\right)\right)R(x)\ dx$$

Probability

$$\sum_{i=3}^{6}\pi_i(\kappa,t)=1$$

Generalized Beta prime distribution

$${\rm B}'(x;\alpha,\beta,p,q)=\frac{p}{qB(\alpha,\beta)}\left(\frac{x}{q}\right)^{\alpha p-1}\left(1+\left(\frac{x}{q}\right)^{p}\right)^{-(\alpha+\beta)}$$

Rejection function

$$R(x;a,b)\equiv\frac{(1+x)\sqrt{x+2}}{\sqrt{2}+ax^{1/2}+b\sqrt{2}x+x^{3/2}}$$

(2) 棄却法の採択効率を
70%→95%に改善
（Maxwell-Jüttner分布の生成効率も向上）

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