9 Rotation and hydrodynamics

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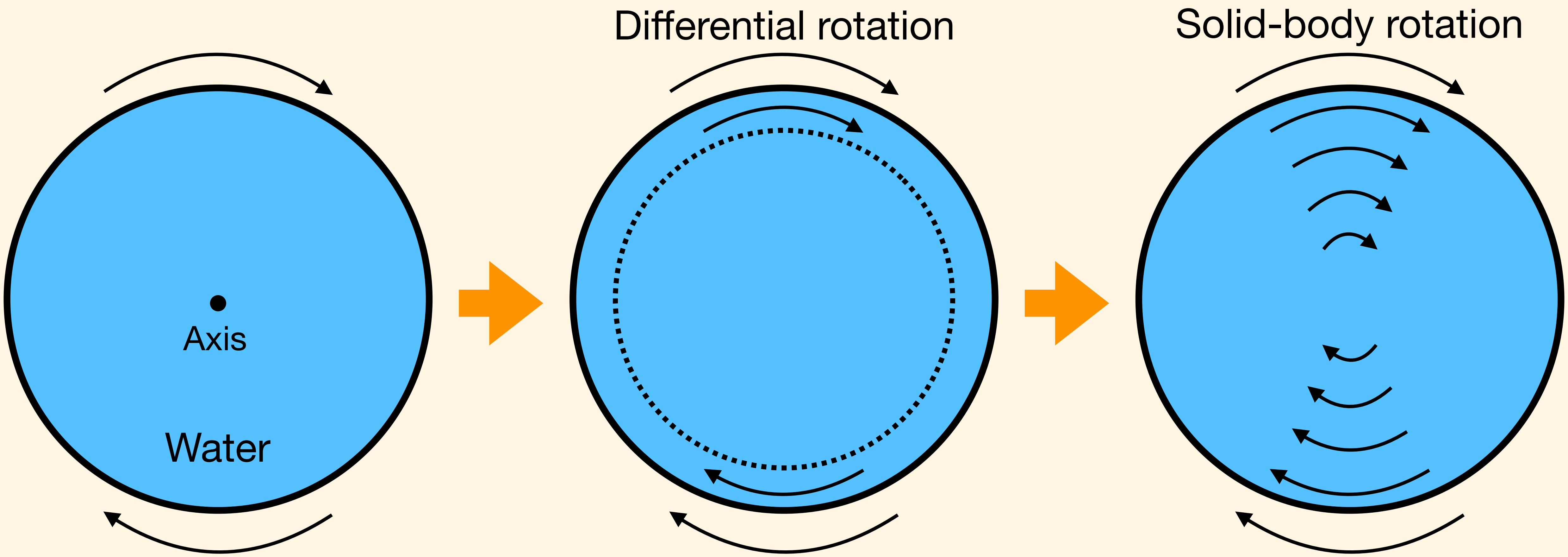
Water in a bucket



- We take water in a bucket and start rotating the bucket around its axis.

- Only the water near the walls of the bucket starts rotating with the bucket.
- The water in the central part still remains at rest.
- **Angular velocity varies within the water.**

- Viscosity eventually stops the relative motions amongst different layers of water.
- **The whole water spins with the same angular velocity as that of bucket.**

Navier-Stoks equation in cylindrical coordinates

- In the astrophysical Universe, many objects rotating not like a solid body with two reasons.
  1. Viscous forces may not have enough time to establish a solid-body rotation.
  2. There may be some physical mechanism which maintains the differential rotation.

- Consider how centrifugal force in a rotation body of fluid is balanced to understand more.

  ⟶ **Navier-Stoks equation in cylindrical coordinates**

$$\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_z \frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} + \frac{\partial^2 v_r}{\partial z^2} + \frac{1}{r}\frac{\partial v_r}{\partial r} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} - \frac{v_r}{r^2}\right) + F_r$$

$$\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + v_z \frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + \nu\left(\frac{\partial^2 v_\theta}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\partial^2 v_\theta}{\partial z^2} + \frac{1}{r}\frac{\partial v_\theta}{\partial r} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r^2}\right) + F_\theta$$

$$\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2} + \frac{1}{r}\frac{\partial v_z}{\partial r}\right) + F_z$$

Accretion disks or the disk of a spiral galaxy

Navier-Stoks equation in cylindrical coordinates

$$\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_z \frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} + \frac{\partial^2 v_r}{\partial z^2} + \frac{1}{r}\frac{\partial v_r}{\partial r} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} - \frac{v_r}{r^2}\right) + F_r$$

A steady, axisymmetric rotation such that we can put $\frac{\partial}{\partial t} = 0,\ \frac{\partial}{\partial \theta} = 0,\ v_r = 0$

$\longrightarrow$ $-\frac{v_\theta^2}{r} = g_r - \frac{1}{\rho}\frac{\partial p}{\partial r} \quad (9.1)$ $\quad \because g_r$ is the r component of the gravitational field.

① In some astrophysical system such as accretion disk or disk of a spiral galaxy, **the pressure gradient force is not important.**

② **The centrifugal force is balanced by gravity** so the angular velocity is given by $v_\theta / r = \sqrt{|g_r|/r}$.

③ The effect of viscosity is to induce a slow radial inflow of matter rather than to produce solid-body rotation.

The effect of viscosity in a thin accretion disk

The dynamics of a thin accretion disk using cylindrical coordinates $\left(v_z = 0,\ \ \partial/\partial\theta = 0\right)$

Continuity equation

$$\frac{\partial\rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\rho v_r) = 0 \quad (5.41)$$

Navier-Stokes equation

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_r v_\theta}{r}\right) = \rho\nu\left(\frac{\partial^2 v_\theta}{\partial r^2} + \frac{\partial^2 v_\theta}{\partial z^2} + \frac{1}{r}\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r^2}\right) \quad (5.42)$$

$\because \int \rho dz = \Sigma$: surface density

$$\frac{\partial\Sigma}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\Sigma v_r) = 0 \quad (5.43)$$

$$\Sigma\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_r v_\theta}{r}\right) = \text{terms involving viscosity} \quad (5.44)$$

where we neglect the variation of v_r and v_θ with z (although ρ varies with z , we do not expect the components of velocity to vary much).

$(5.43) \times r v_\theta + (5.44) \times r$

$$\frac{\partial}{\partial t}(\Sigma r^2\Omega) + \frac{1}{r}\frac{\partial}{\partial r}(\Sigma r^3\Omega v_r) = \text{terms involving viscosity} \quad (5.45)$$

$$\because \Omega = v_\theta / r$$

$\Sigma r^2\Omega \cdot 2\pi r\ dr$

: the angular momentum associated with an annular ring from between r and $r + dr$.

$\frac{1}{r}\frac{\partial}{\partial r}(\Sigma r^3\Omega v_r)$

: the divergence of the angular momentum flux $\Sigma r^2\Omega v_r \mathbf{e_r}$ due to the radial flow.

$$\because \Sigma r^3\Omega v_r = r \cdot \Sigma r^2\Omega \cdot v_r$$

$$\because \text{div } A = \frac{1}{r}\left[\frac{\partial\,(rA_r)}{\partial r} + \frac{\partial\,(A_\phi)}{\partial\phi} + \frac{\partial\,(A_z)}{\partial z}\right]$$

(5.45) : the evolution of the angular momentum

See also 5.7.1 The basic disk dynamics

Inside a slowly rotating star

① The gravitational field and the pressure gradient nearly balance each other.

② There remains an unbalanced part of pressure gradient and it counteracts the centrifugal force.

③ The stars seem to rotate like solid bodies due to the effect of viscosity because there is no obvious mechanism to sustain differential rotation in such stars.

Viscosity

① The effect of molecular viscosity inside the star is negligible.

② Viscosity appears in the Navier-Stokes equation in the term of $\nu \nabla^2 \mathbf{v}$, which is not significant in systems with large scale.

Navier-Stokes equation : $$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{F} - \frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{v} \quad (5.10)$$

Convection

① Convection inside the star occurs the turbulent diffusion of various quantities like the angular momentum (cf. 8.4 Turbulent diffusion).

② It is sometimes appropriate to introduce an anisotropic viscosity to model the turbulent diffusion (Kippenhahn 1963).

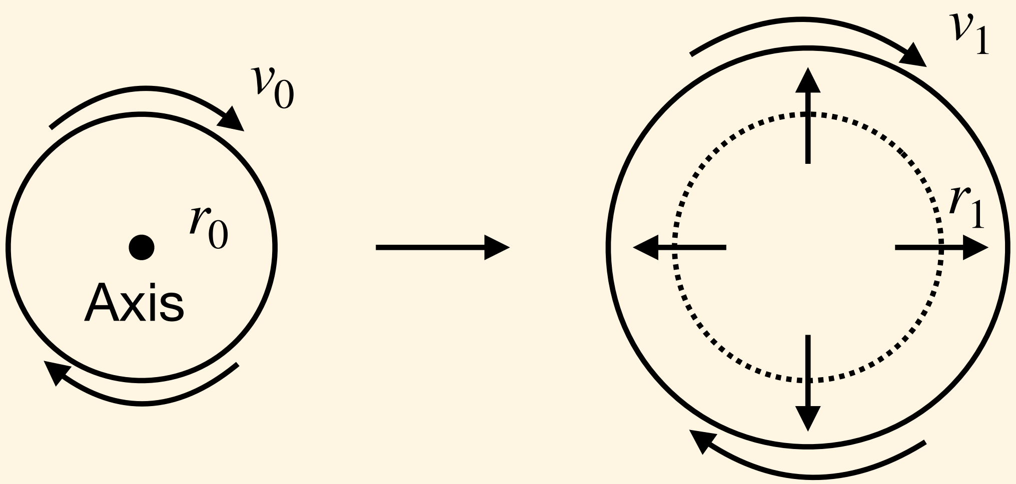
③ The steady state with anisotropic viscosity is a state of differential rotation.

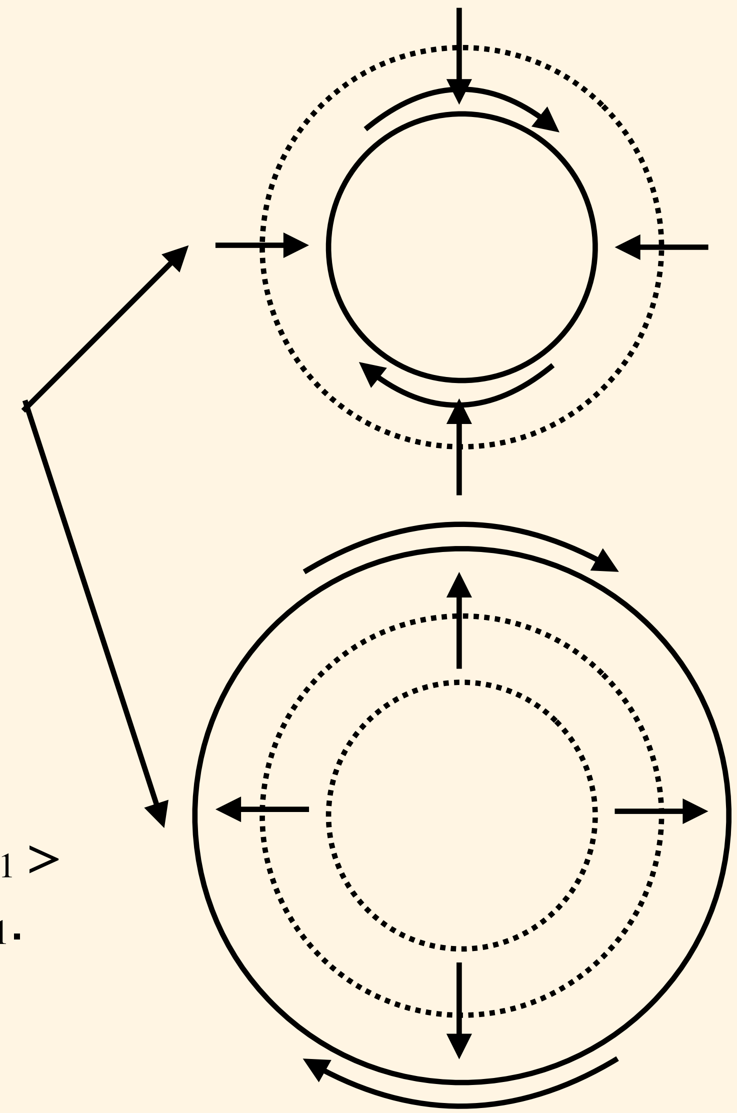
④ If the viscosity is made more isotropic, the star rotates more like a solid-body.

In the sun, the angular velocity between the pole and the equator differs by more than 10%.

Rayleigh's criterion

All types of differential rotation are not stable.





Stable

The rings tend to return to their initial position.

Unstable

The rings move further away.

A fluid ring at a distance r_0 from the axis moving the velocity v_0.

The ring is interchanged with a fluid ring at a greater distance r_1 (i.e. $r_1 > r_0$) moving the velocity v_1.

Assume the conservation of angular momentum.

calculation : next page

Rayleigh's criterion

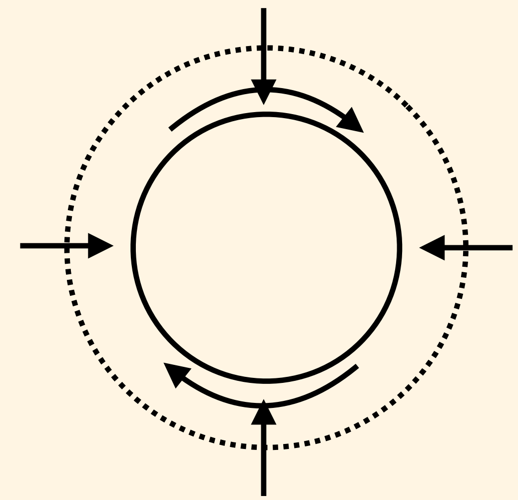
- The displaced ring acquires a velocity $\frac{r_0}{r_1}v_0$.
- The ring previously at r_1 had a centripetal acceleration $\frac{v_1^2}{r_1}$.

  → It has been provided by the various forces there such as the part of the pressure gradient left after balancing gravity.
- The displaced ring requires a centripetal acceleration $\frac{1}{r_1}\left(\frac{r_0}{r_1}v_0\right)^2 = \frac{r_0^2 v_0^2}{r_1^3}$ to remain in its new position.
- If $\frac{r_0^2 v_0^2}{r_1^3} < \frac{v_1^2}{r_1}$, the forces present there will push the ring inward towards its initial position.

The condition for stability is $\frac{r_0^2 v_0^2}{r_1^3} < \frac{v_1^2}{r_1}$

→ $(r_0^2\Omega_0)^2 < (r_1^2\Omega_1)^2 \quad \because \Omega_0 = v_0 r_0, \Omega_1 = v_1 r_1$

$\therefore \frac{d}{dr}[(r_0^2\Omega_0)^2] > 0 \quad (9.2)$: Rayleigh's criterion



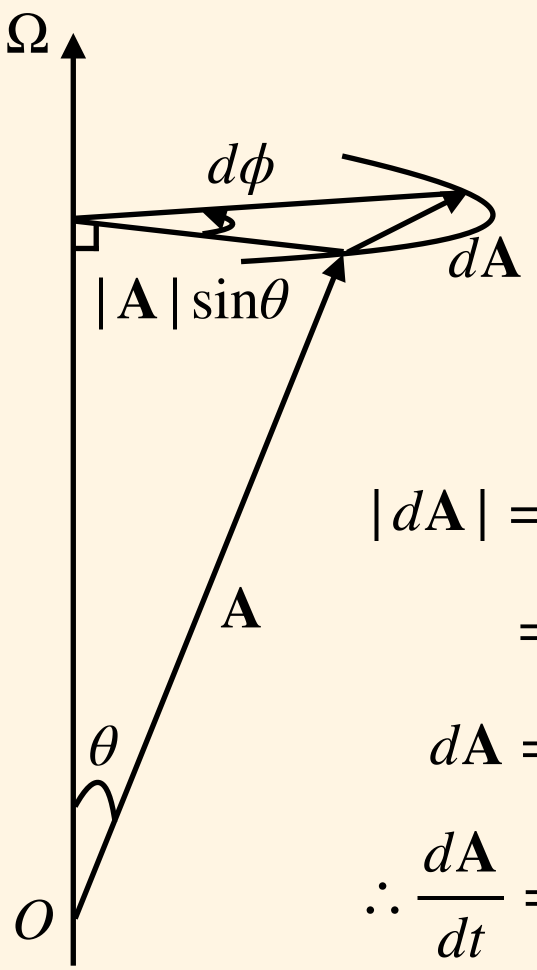
Stable

The rings tend to return to their initial position.

Acceleration in the rotating frame

Inertial frame $\mathbf{r} = (x, y, z)$

Rotating reference frame $\mathbf{r}' = (x', y', z')$



$$|d\mathbf{A}| = |\mathbf{A}|\sin\theta \times d\phi$$

$$= |\mathbf{A}|\sin\theta \times |\boldsymbol{\Omega}|\,dt$$

$$d\mathbf{A} = \boldsymbol{\Omega} \times \mathbf{A}dt$$

$$\therefore \frac{d\mathbf{A}}{dt} = \Omega \times A$$

$$\mathbf{r} = \mathbf{r}' = x'\mathbf{e}'_x + y'\mathbf{e}'_y + z'\mathbf{e}'_z$$

$$\longrightarrow \quad \frac{d\mathbf{r}}{dt} = \mathbf{v}' + \boldsymbol{\Omega} \times \mathbf{r}'$$

$$\left(\because \frac{d\mathbf{e}_i}{dt} = \Omega \times \mathbf{e}_i \right)$$

$$\longrightarrow \quad \frac{d^2\mathbf{r}}{dt^2} = \frac{d^2\mathbf{r}'}{dt^2} + 2\Omega \times \mathbf{v}' + \frac{d\boldsymbol{\Omega}}{dt} \times \mathbf{r}' + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}')$$

In a frame of reference rotating with constant angular velocity $\boldsymbol{\Omega}$

$$\frac{d\mathbf{v}}{dt} \longrightarrow \frac{d\mathbf{v}}{dt} + 2\Omega \times \mathbf{v}' + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}') \quad (9.3)$$

Navier-Stokes equation in the rotating frame

- Making the replacement is the Navier-Stokes equation, we obtain the equation of motion in the rotating frame

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{F} - \frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{v} \quad (5.10)$$

Coriolis force

$$\longrightarrow \quad \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{F} - \frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{v} - 2\mathbf{\Omega} \times \mathbf{v} - \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) \quad (9.4)$$

Centrifugal force, which can also be written as $\frac{1}{2}\nabla(|\mathbf{\Omega} \times \mathbf{r}|^2)$

- If the body force $\mathbf{F}$ is of gravitational origin, we can write it as $-\nabla\Phi$, where Φ is the gravitational potential.

$$\longrightarrow \quad \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p - \nabla\left(\Phi - \frac{1}{2}(|\mathbf{\Omega} \times \mathbf{r}|^2)\right) + \nu \nabla^2 \mathbf{v} - 2\mathbf{\Omega} \times \mathbf{v} \quad (9.5)$$

The basic equation of motion for fluids in a rotating frame of reference

① The effect of the centrifugal force is to introduce a potential force that modifies gravity.

② An effective gravitational potential is introduce as follows.

$$\Phi_{\mathrm{eff}} = \Phi - \frac{1}{2}(|\mathbf{\Omega} \times \mathbf{r}|^2) \quad (9.6)$$

Coriolis force

fluid flows through pipes in the laboratory	⟷	large ocean currents and monsoon winds
We do not have to concern that the Earth is a rotating frame.		They are influenced by the Earth rotation very much.

⬇

To figure out how important the Coriolis force is, let's compare $2\mathbf{\Omega}\times\mathbf{v}$ with $(\mathbf{v}\cdot\nabla)\mathbf{v}$ in (9.5).

※Coriolis force is non-zero only when there are motions with respect to the rotating frame.

If V and L are the typical velocity and length scales,

$(\mathbf{v}\cdot\nabla)\mathbf{v}$ is order of V^2/L

Coriolis force $-2\mathbf{\Omega}\times\mathbf{v}$ is order of ΩV

The ratio of the two is

$$\epsilon = \frac{V^2/L}{\Omega V} = \frac{V}{\Omega L} \qquad (9.7)$$

Rossby number

The Coriolis force is important if the Rossby munger is of order unity or less.

⟶ This is the case for large-scale motions in the atmosphere or the oceans, but not for most fluid phenomena in the laboratory.

Large-scale atmospheric or oceanic circulations

① The atmosphere or the ocean is much thinner than the dimensions of the Earth.

- The fluid flows are nearly horizontal.

② The flows are changing slowly and have low Rossby numbers.

- The terms on the L.H.S. of (9.5) are very small compare to the terms on the R.H.S.

$$-\frac{\nabla p}{\rho} - g\mathbf{e}_{\mathrm{r}} - 2\boldsymbol{\Omega} \times \mathbf{v} = 0 \quad (9.8) \qquad \because -\nabla \Phi = -g\mathbf{e}_{\mathrm{r}}$$

③ The magnitude of Coriolis force is small compare to the gravity.

- The Coriolis force does not play an important role in the force balance in the vertical direction.

$$\frac{1}{\rho}\frac{\partial p}{\partial r} = -g \quad (9.9)$$

: vertical direction

④ Gravity is absent in the horizontal direction.

- Coriolis force gets the chance to play a dominant role in the horizontal force balance.

$$\nabla_{\mathrm{h}} p = -2\rho(\boldsymbol{\Omega} \times \mathbf{v})_{\mathrm{h}} \quad (9.10)$$

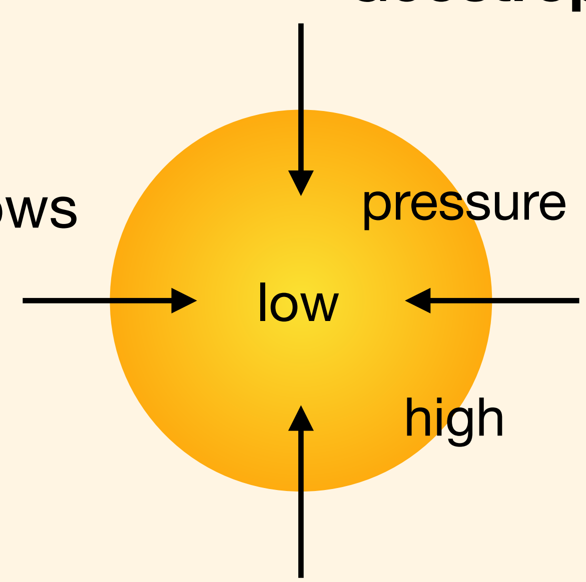
: horizontal direction

(h implies the horizontal components)

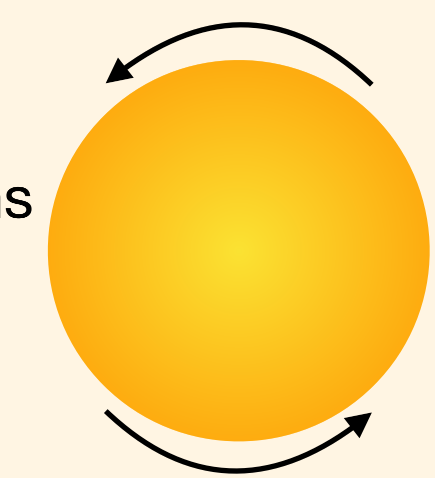
Geostrophic approximation (地衡流近似)

Depression of pressure

We expect the velocity of flows $\mathbf{v}$ in the direction of $-\nabla p$.



In geophysical circulations



Equation of motion with vorticity

We will consider ideal, incompressible fluids. $\longrightarrow$ $\nu = 0, \quad \nabla \cdot \mathbf{v} = 0 \quad (\rho = \text{constant})$

$$(9.5) \quad \& \quad (\mathbf{v} \cdot \nabla)\mathbf{v} = \frac{1}{2}\nabla(\mathbf{v} \cdot \mathbf{v}) - \mathbf{v} \times (\nabla \times \mathbf{v}) \quad (4.16) \quad \& \quad \omega = \nabla \times \mathbf{v} \quad (4.18)$$

$$\longrightarrow \quad \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \times \omega = -\nabla\left(\frac{p}{\rho} + \frac{1}{2}v^2 + \Phi - \frac{1}{2}|\mathbf{\Omega} \times \mathbf{r}|^2\right) - 2\mathbf{\Omega} \times \mathbf{v} \quad (9.11)$$

taking the curl

$$\frac{\partial \omega}{\partial t} = \nabla \times (\mathbf{v} \times \omega) + \nabla \times (\mathbf{v} \times 2\mathbf{\Omega})$$

Since $\mathbf{\Omega}$ is constant in time, $\frac{\partial}{\partial t}(\omega + 2\mathbf{\Omega}) = \nabla \times [\mathbf{v} \times (\omega + 2\mathbf{\Omega})] \quad (9.12)$

$$\frac{d}{dt}\int_S (\omega + 2\mathbf{\Omega}) \cdot d\mathbf{S} = 0 \quad (9.13)$$

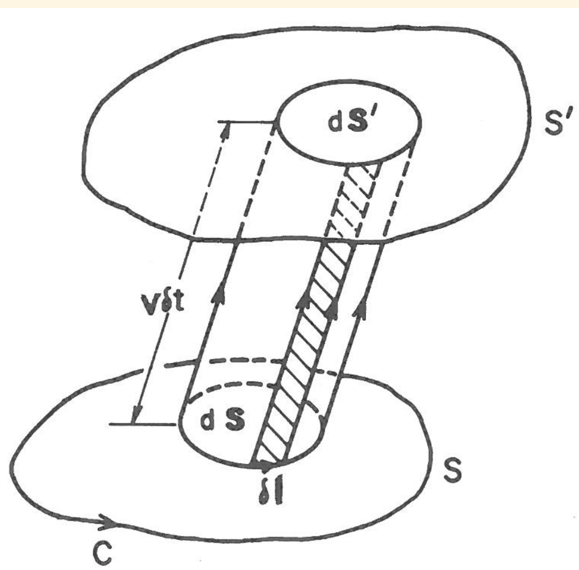
Kelvin's vorticity theorem

$$\frac{\partial \mathbf{Q}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{Q}) \quad (4.42) \longrightarrow \frac{d}{dt}\int_S \mathbf{Q} \cdot d\mathbf{S} = 0 \quad (4.43)$$

$\because$ $\mathbf{Q}$ is any vector field

The generalization of Kelvin's vorticity theorem in rotting frames(Bjerknes's theorem)

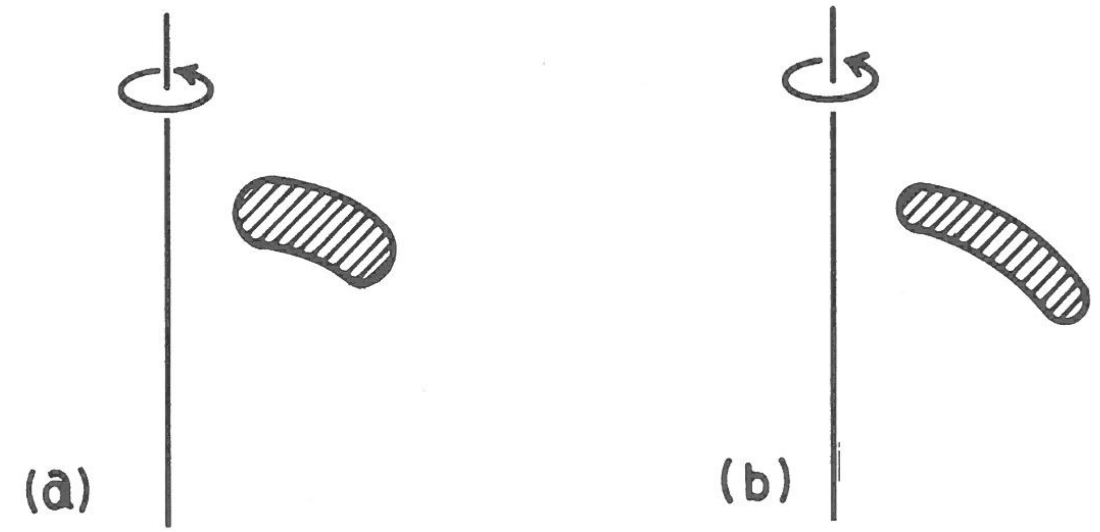
Figure 4.4
Displacement of a surface element due to fluid motions.



Generations of vortices in rotating frames

A volume fluid which is at rest in a rotating frame(Figure 9.1(a)).

Figure 9.1 The spreading of a volume of fluid into a thinner layer in a rotating frame of reference.



The fluid is squeezed to spread into a thinner layer(Figure 9.1(b)).

- The flux $\int_S \mathbf{\Omega} \cdot d\mathbf{S}$ associated with the fluid becomes larger.
- Vorticity ω is developed opposite to $\mathbf{\Omega}$ so that the integral $\int_S (\omega + 2\mathbf{\Omega}) \cdot d\mathbf{S}$ remains conserved even though there was no vorticity initially.

⟶ This gives an idea how cyclonic storms may be produced in the Earth's atmosphere.

Taylor-Proudman theorem

We will consider ideal, incompressible fluids. ⟶ $\nu = 0, \quad \nabla \cdot \mathbf{v} = 0 \quad (\rho = \text{constant})$

For steady fluid in a rotating frame of reference,

$$\nabla \times [\mathbf{v} \times (\omega + 2\boldsymbol{\Omega})] = 0 \qquad \left(\because \frac{\partial}{\partial t}(\omega + 2\boldsymbol{\Omega}) = 0 \text{ in } (9.12) \right)$$

If the flows are slow such that any vorticity associated with the flow is small compare to $\boldsymbol{\Omega}$,

$$\nabla \times (\mathbf{v} \times 2\boldsymbol{\Omega}) = 0 \quad (9.14) \qquad \longrightarrow \qquad (\boldsymbol{\Omega} \cdot \nabla)\mathbf{v} = 0 \quad (9.15)$$

$$(\because \nabla \times (\mathbf{v} \times 2\boldsymbol{\Omega}) = (\boldsymbol{\Omega} \cdot \nabla)\mathbf{v} - (\mathbf{v} \cdot \nabla)\boldsymbol{\Omega} + \mathbf{v}(\nabla \cdot \boldsymbol{\Omega}) - \boldsymbol{\Omega}(\nabla \cdot \mathbf{v}))$$

- $\mathbf{v}$ does not change in the direction of $\boldsymbol{\Omega}$.
- Slow steady flows in rotating frames tend to be invariant parallel to the rotation axis,

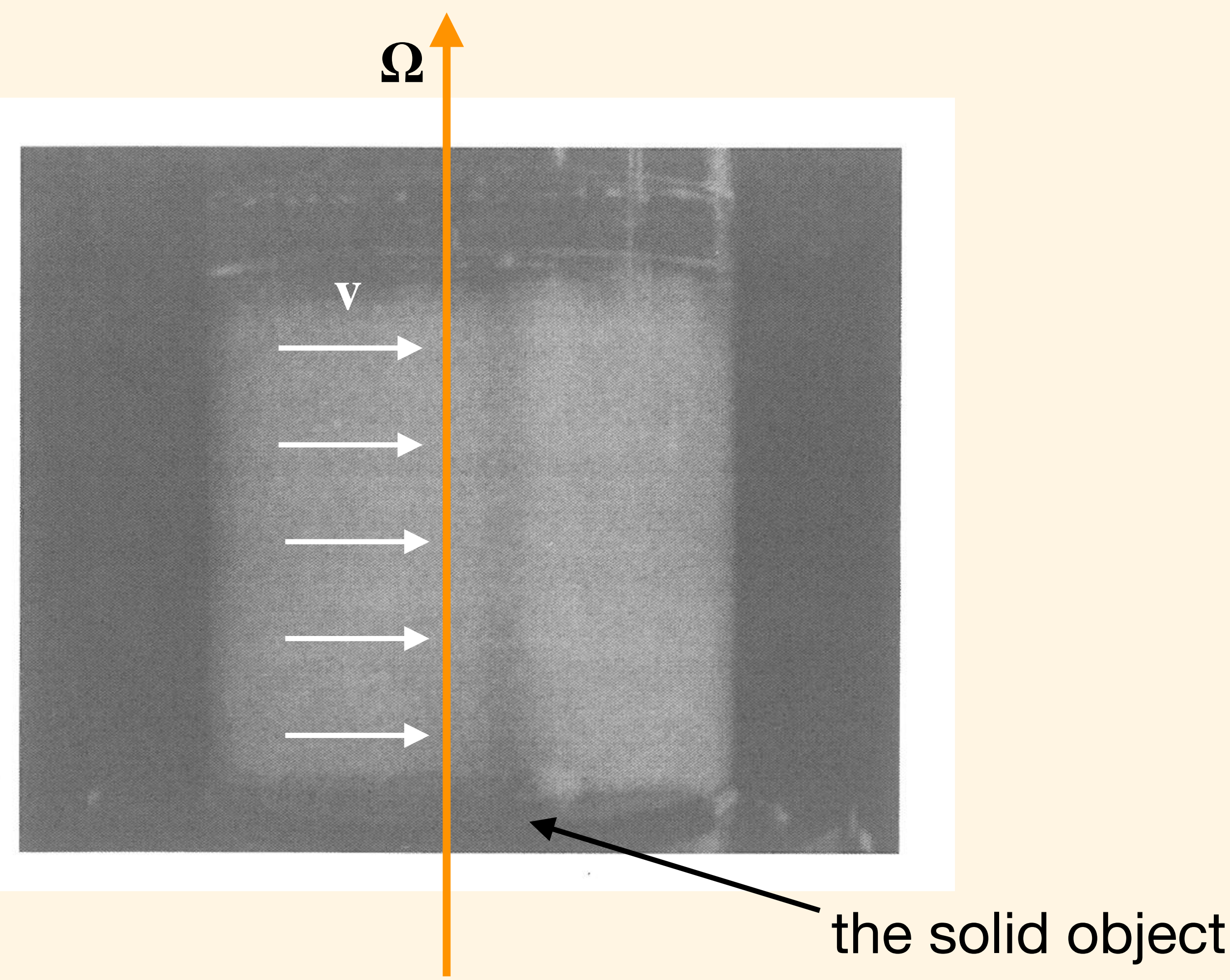


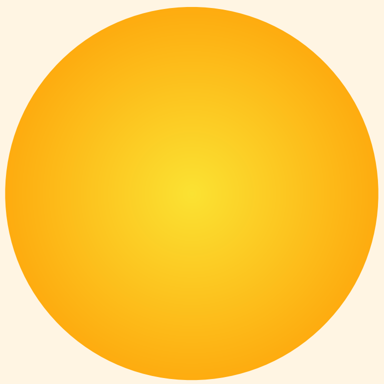
Figure 9.2 A photograph showing a Taylor–Proudman column in a rotating fluid. Reproduced from Greenspan (1968). (©Cambridge University Press.)

Self-gravitating rotating fluid mass

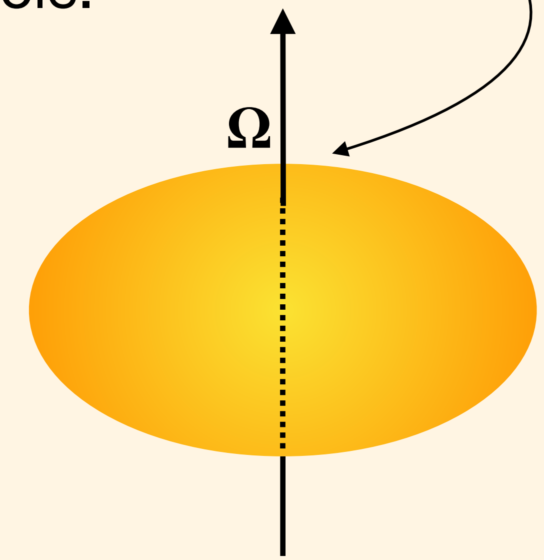
constant density ρ

If there is no rotation in the system, the equilibrium configuration of the system is spherical.

The rotation cause some flattening near the pole.



→ Put in some angular momentum



- In a frame with angular velocity $\mathbf{\Omega}$ in which the fluid is everywhere at rest (solid-body rotation).

from (9.11), $\nabla\left(\frac{p}{\rho} + \Phi - \frac{1}{2}|\mathbf{\Omega}\times\mathbf{r}|^2\right) = 0$ $\quad\therefore \frac{p}{\rho} + \Phi - \frac{1}{2}|\mathbf{\Omega}\times\mathbf{r}|^2 = \text{constant} \quad (9.16)$

- On the outer surface of the fluid mass, the pressure can be taken to be zero.
- Choosing the z axis along the axis of rotation, we have from (9.16) that

$$\Phi - \frac{1}{2}\Omega^2(x^2 + y^2) = \text{constant} \quad (9.17)$$

We want to establish that the fluid takes up the shape of an ellipsoid.

↓

We have to obtain the gravitational potential Φ due to an ellipsoid.

Gravitational potential due to the ellipsoid

An ellipsoid with uniform density ρ inside the boundary surface

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \quad (9.18)$$

The gravitational potential at any point inside this ellipsoid is given by

$$\Phi=\pi G\rho(\alpha_0 x^2+\beta_0 y^2+\gamma_0 z^2-\chi_0) \quad (9.19)$$

where

$$\alpha_0=abc\int_0^\infty \frac{d\lambda}{(a^2+\lambda)\Delta}, \quad \beta_0=abc\int_0^\infty \frac{d\lambda}{(b^2+\lambda)\Delta}, \quad \gamma_0=abc\int_0^\infty \frac{d\lambda}{(c^2+\lambda)\Delta} \quad (9.20)$$

and

$$\chi_0=abc\int_0^\infty \frac{d\lambda}{\Delta} \quad (9.21)$$

Only the results

Δ being given by

$$\Delta=[(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)]^{1/2} \quad (9.22)$$

See also Chandrasekhar (1969 Chapter 3)

If the rotating mass of fluid takes up the shape of an ellipsoid, (9.19) at the bounding surface has to satisfy (9.17).

$$\left(\alpha_0-\frac{\Omega^2}{2\pi G\rho}\right)x^2+\left(\beta_0-\frac{\Omega^2}{2\pi G\rho}\right)y^2+\gamma_0 z^2=\text{constant} \quad (9.23)$$

In order fo (9.18) and (9.23) to hold simultaneously,

$$\left(\alpha_0-\frac{\Omega^2}{2\pi G\rho}\right)a^2=\left(\beta_0-\frac{\Omega^2}{2\pi G\rho}\right)b^2=\gamma_0 c^2 \quad (9.24)$$

➡ The shape of the fluid : (a, b, c)

Maclaurin spheroids and Jacobi ellipsoids

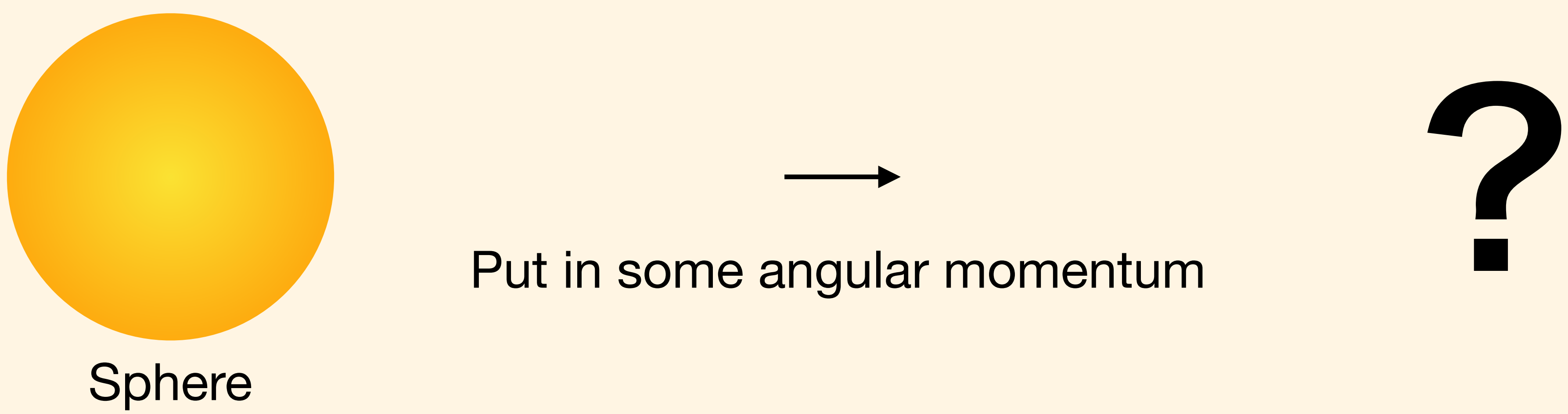
Two solutions of (9.24);

① **Maclaurin spheroids**(9.3.1)

- The rotating fluid takes up a symmetric and flattened configuration around the rotation axis.

② **Jacobi ellipsoids**(9.3.2)

- The rotating fluids have three unequal axes under certain circumstance.



$$\left(\alpha_0 - \frac{\Omega^2}{2\pi G\rho}\right)a^2 = \left(\beta_0 - \frac{\Omega^2}{2\pi G\rho}\right)b^2 = \gamma_0 c^2 \quad (9.24)$$

Maclaurin spheroids

The rotating fluid takes up a symmetric and flattened configuration around the rotation axis.

$$a = b > c \quad (9.25) \qquad \because \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad (9.18)$$

An ellipsoid with such axes is called a spheroid and the eccentricity （離心率） is defined by

$$e^2 = 1 - \frac{c^2}{a^2} \quad (9.26)$$

The eccentricity e
- $e = 0$: circle
- $0 < e < 1$: ellipse
- $e = 1$: parabola
- $1 < e$: hyperbola

(9.20) ⟶

$$\alpha_0 = \beta_0 = \frac{(1-e^2)^{1/2}}{e^3}\sin^{-1}e - \frac{1-e^2}{e^2} \quad (9.27)$$

$$\gamma_0 = \frac{2}{e^2}\left[1 - (1-e^2)^{1/2}\frac{\sin^{-1}e}{e^2}\right] \quad (9.28)$$

(9.27), (9.28) ⟶ (9.24) ⟶ (9.29)

The relation between Ω and e is given by

$$\frac{\Omega^2}{\pi G \rho} = \frac{2(1-e^2)^{1/2}}{e^3}(3-2e^2)\sin^{-1}e - \frac{6}{e^2}(1-e^2) \quad (9.29)$$

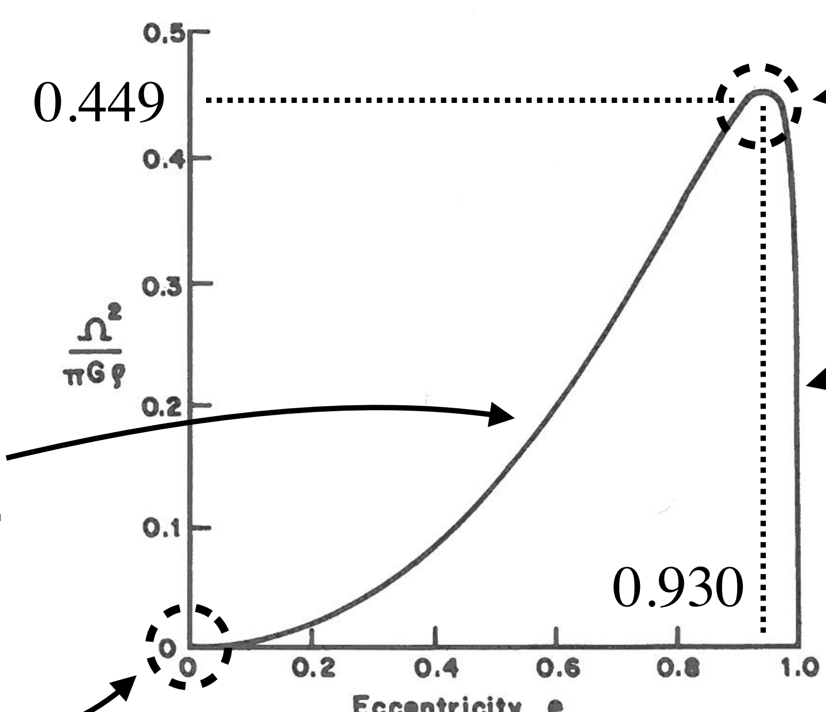


Figure 9.3 The relation between $\Omega^2/\pi G\rho$ and eccentricity e of Maclaurin spheroids. Adapted from Chandrasekhar (1969).

$\Omega^2/\pi G\rho$ reaches a maximum

A spheroid becomes more flattened with more rotation.

Less rotation makes the system more flattened.

A spherical configuration (e=0) corresponding to no rotation (Ω=0).

Maclaurin spheroids

The angular momentum of the spheroid is

$$L = \frac{2}{5}Ma^2\Omega \quad (9.30) \qquad \because \text{the moment of inertia of the spheroid } I = \frac{2}{5}Ma$$

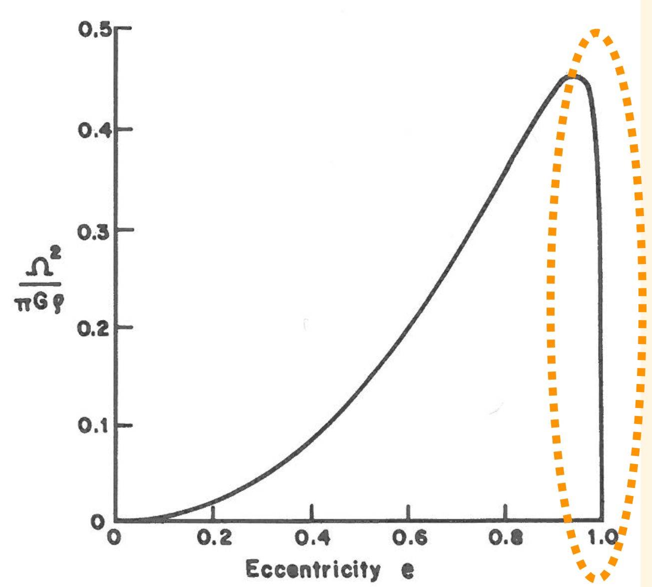
$$\because M = \frac{4}{3}\pi a^2 c\rho = \frac{4}{3}\pi a^3(1-e^2)^{1/2}\rho \quad (9.31)$$

The dimension less angular momentum is

$$\overline{L} = \frac{L}{[GM^3(a^2c)^{1/3}]^{1/2}} = \frac{2\sqrt{3}}{5}\left(\frac{a^{2/3}}{c}\right)\left(\frac{\Omega^2}{\pi G\rho}\right)^{1/2} \quad (9.32)$$

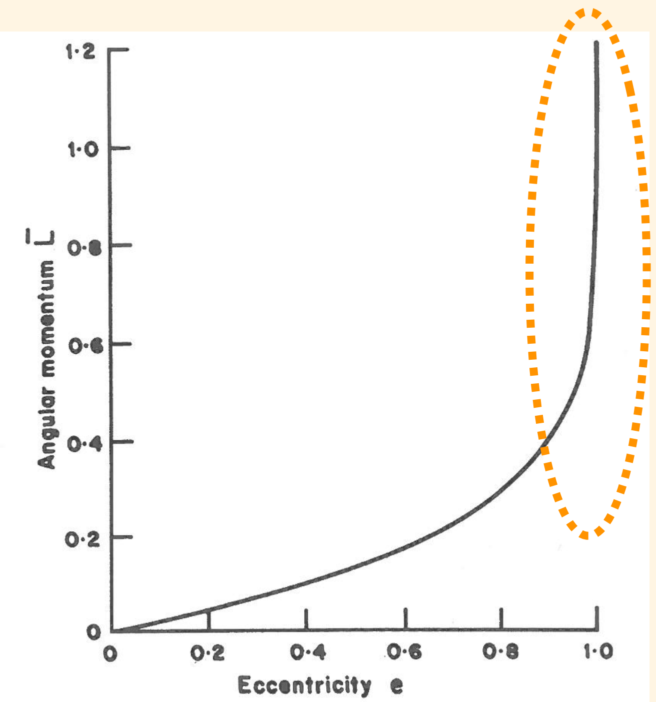
: constant for a given mass of rotating fluids

Figure 9.3 The relation between $\Omega^2/\pi G\rho$ and eccentricity e of Maclaurin spheroids. Adapted from Chandrasekhar (1969).



The relation between Ω and e

Figure 9.4 The relation between dimensionless angular momentum $\overline{L}$ and eccentricity e of Maclaurin spheroids. Adapted from Chandrasekhar (1969).



The relation between $\overline{L}$ and e

The additional angular momentum makes the system so flattened and increases the momentum of inertia at a such fast rate.

The angular velocity goes down

Jacobi ellipsoids

Ellipsoidal solutions with three unequal axes under certain circumstance.

(9.24) ⟶ $(\alpha_0 - \beta_0)a^2b^2 + \gamma_0 c^2(a^2 - b^2) = 0 \quad (9.33)$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad (9.18)$$

$$\alpha_0 = abc\int_0^\infty \frac{d\lambda}{(a^2+\lambda)\Delta}, \quad \beta_0 = abc\int_0^\infty \frac{d\lambda}{(b^2+\lambda)\Delta}, \quad \gamma_0 = abc\int_0^\infty \frac{d\lambda}{(c^2+\lambda)\Delta} \quad (9.20)$$

$$(a^2 - b^2)\int_0^\infty \left[\frac{a^2b^2}{(a^2+\lambda)(b^2+\lambda)} - \frac{c^2}{c^2+\lambda}\right]\frac{d\lambda}{\Delta} = 0 \quad (9.34)$$

(9.34) is the result of calculation so to demonstrate the existence of Jacobi ellipsoids, we need to find real and unequal values of (a, b, c).

⟶ We will only see the results.

Maclaurin spheroids or Jacobi ellipsoids

Two solutions of (9.24);

① **Maclaurin spheroids**[9.3.1]

- The rotating fluid takes up a symmetric and flattened configuration around the rotation axis.

② **Jacobi ellipsoids**[9.3.2]

- The rotating fluids have three unequal axes under certain circumstance.

I. If the dimensionless angular momentum $\bar{L}$ is less than 0.304(corresponding to $e < 0.813$)

Maclaurin spheroids are the only possible solution.

II. If the dimensionless angular momentum $\bar{L}$ is larger than 0.304

There are two solution: a Maclaurin spheroid and a Jacobi ellipsoid.

- For given angular momentum, the Jacobi ellipsoid has less rotational kinetic energy compare to the Maclaurin spheroid.
- → If there is some dissipation mechanism like viscosity, the Maclaurin spheroid becomes unstable.
- It will relax to a Jacobi ellipsoid after some dissipation of energy.

video(Jacobi's ellipsoidal Earth) : https://www.dailymotion.com/video/x2rv2h1

We considered the solid-body rotation of a mass of incompressible fluid.

Rotation in the world of stars

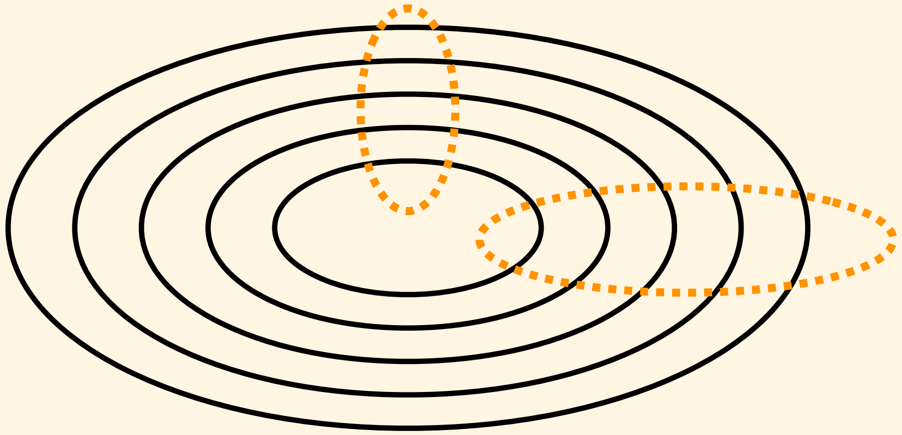
We assume that a compressible star can be completely at rest in the rotating frame of reference.

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p - \nabla\left(\Phi - \frac{1}{2}(|\mathbf{\Omega} \times \mathbf{r}|^2)\right) + \nu \nabla^2 \mathbf{v} - 2\mathbf{\Omega} \times \mathbf{v} \quad (9.5)$$

$$\longrightarrow \quad \frac{\nabla p}{\rho} = -\nabla \Phi_{\mathrm{eff}} \quad (9.33) \quad \xrightarrow[\nabla \times]{} \quad \nabla p \times \nabla \rho = 0 \ \ ? \qquad \nabla p \times \nabla\left(\frac{1}{\rho}\right) = 0 \ \ ?$$

1. The contours of constant p and constant ρ are coincide.
2. From (9.33), the contours should be the contour of constant Φ_{eff}.
3. From the perfect gas law, the contours also should be the contour of constant T.

Because of the spheroidal shapes of these contours, the temperature gradient near the pole of a particular contour is larger than that near the equator.



The image of the shapes of contours

→ The radiative energy flux coming through the poles of a star is larger than that coming through the equator.

Rotation in the world of stars

- **Stellar models neglecting rotation give reasonably good results.**
  - In the case of the Sun and the main-sequence stars, the rotation is not too strong.
  - The centrifugal force is a small fraction of gravity.
- **Rotation becomes important in collapsed stars.**

When we decrease the radius a of a star, if the angular momentum is conserved,

- The moment of inertia decrease as a^2.
- The angular velocity Ω goes as a^{-2}.
- The centrifugal force $\Omega^2 a$ goes as a^{-3}.
- The gravity increase as a^{-2}.

The centrifugal force increase faster with decreasing radius than gravity.

A maximum limit of the rotation rate for a mass of incompressible fluid is(see section 9.3.1)

$$\Omega^2 < 0.449\pi G\rho \quad (9.36)$$

The fluid mass would fall apart if it is made to rotate faster(The corresponding condition for the period is $T=2\pi/\Omega$).

$$T > \frac{2.05\times 10^4}{\sqrt{\rho}} \quad (9.37)$$

<u>The rotation is also important during the star's birth. We need some mechanisms for removing the angular momentum from the dense cores in order to form stars.</u>

Rotation in the world of galaxies

We should apply the equations of stellar dynamics rather than the equations of hydrodynamics to study galactic rotation. It is(see Exercise 3.3)

$$\frac{\partial}{\partial r}(n\langle\Pi^2\rangle) + \frac{\partial}{\partial z}(n\langle\Pi Z\rangle) + \frac{n}{r}[\langle\Pi^2\rangle - \langle\Theta^2\rangle] = ng_r \quad (9.38)$$

where Π, Θ and Z are the r, θ and z components of the velocities of stars, which have the number density n.

Let's consider the situation in which all stars move in regular circular orbits without any random motion.

$$\because \Pi = Z = 0 \longrightarrow \Theta = \Theta_c = \sqrt{r|g_r|} \quad (9.39)$$

(the gravity is completely balanced by the centrifugal force.)

If the stars have random velocities, then the gravity can be partially or fully balanced by the 'pressure' forces arising out of these random motions.

$$\langle\Theta\rangle < \Theta_c$$

(only a part of the gravitational force remains to balanced by the centrifugal force.)

Rotation in the world of galaxies

By analyzing the random motions of stars in the solar neighborhood, theses stars can be divided into two categories;

1. Stars with low Radom velocities which move in nearly circular orbit
2. Stars with high random velocities

For the stars in category 2, they have a much less average rotation around the galactic center compare to the stars in category 1.

⟶ Their random motion balance the gravity and they need not have as much rotation as the stars in category 1.

※ Even within category 1, some groups of stars have more velocity dispersion compare to the others. They seem to lag behind the other stars while going around the galactic center.

The elliptical galaxies have much less rotation compare to the spiral galaxies and the gravity is balanced mainly by the velocity dispersion.

The rotation patterns of spiral galaxy

In these galaxies, most of the visible stars in the disk move in nearly circular orbits with Θ given by (9.39).

If the gravity drops as r^{-2} in the outer regions of the galaxy, Θ should fall off as $r^{-1/2}$

$\because \Theta = \Theta_c \sqrt{r \cdot r^{-2}} = r^{-1/2}$ Such a fall in rotation velocity has not bee observed.

⟶ These galaxies have large amount of unseen dark matter in regions beyond the visible disk so that the gravitational field does not fall off as rapidly as we expect.