« The physics of fluids and plasmas »

Section 6.5 P 115 Xuzhu

Q1: How to derive Equation 6.36 / 6.38

Basic Equations:

$$\rho_1 V_1 = \rho_2 V_2 \qquad (6.33)$$

$$p_1 + \rho_1 V_1^2 = p_2 + \rho_2 V_2^2 \qquad (6.34)$$

$$\frac{1}{2}V_1^2 + \frac{\gamma p_1}{(\gamma-1)\rho_1} = \frac{1}{2}V_2^2 + \frac{\gamma p_2}{(\gamma-1)\rho_2} \qquad (6.35)$$

(Due to it is difficult to distinguish density ρ and pressure P in handwriting version, so just in case, I replace ρ with A in following process, namely:

$$\begin{cases} A_1 V_1 = A_2 V_2 & ① \\ P_1 + A_1 V_1^2 = P_2 + A_2 V_2^2 & ② \\ \frac{1}{2}V_1^2 + \frac{\gamma P_1}{(\gamma-1)A_1} = \frac{1}{2}V_2^2 + \frac{\gamma P_2}{(\gamma-1)A_2} & ③ \end{cases}$$

also Mach number $M_1 = \frac{V_1}{\sqrt{\gamma P_1 / A_1}}$, namely

$$M_1^2 = \frac{A_1 V_1^2}{\gamma P_1} \qquad ④$$

※ Case 1: **Density** ↑ we would like to have the relation of A_1 and A_2, implying the change of the density after the shock wave. So we have to eliminate P_2, V_2.

$$\xrightarrow{①} V_2 = \frac{A_1}{A_2} V_1$$

$$\xrightarrow{②} P_2 = P_1 + A_1 V_1^2 - A_2 V_2^2 = P_1 + A_1 V_1^2 - A_2 \cdot \left(\frac{A_1}{A_2} V_1\right)^2 = P_1 + A_1 V_1^2 - \frac{A_1^2}{A_2} V_1^2$$

multiply $\boxed{2(\gamma-1)A_1 A_2^2}$ to the both sides at the same time:

$$\rightarrow (\gamma-1) A_1 A_2^2 V_1^2 + 2\gamma P_1 A_2^2 = (\gamma-1) A_1 \boxed{A_2^2 V_2^2} + 2\gamma A_1 \boxed{A_2 P_2}$$

due to $\begin{cases} A_2 V_2 = A_1 V_1 \\ A_2 P_2 = A_2 P_1 + A_1 A_2 V_1^2 - A_1^2 V_1^2 \end{cases}$

$$\rightarrow (\gamma-1) A_1 A_2^2 V_1^2 + 2\gamma P_1 A_2^2 = (\gamma-1) A_1^3 V_1^2 + 2\gamma A_1 (A_2 P_1 + A_1 A_2 V_1^2 - A_1^2 V_1^2)$$

Our goal is to get $\frac{A_2}{A_1}$, so divide the both sides by $\boxed{A_1^2}$ according to the highest order of A_2 is 2:

$$\rightarrow (\gamma-1) A_1 \left(\frac{A_2}{A_1}\right)^2 V_1^2 + 2\gamma P_1 \left(\frac{A_2}{A_1}\right)^2 = (\gamma-1) A_1 V_1^2 + 2\gamma \left(\frac{A_2}{A_1} P_1 + A_1 V_1^2 \cdot \frac{A_2}{A_1} - A_1 V_1^2\right)$$

let $x = \frac{A_2}{A_1}$

$$\rightarrow (\gamma-1) A_1 V_1^2 x^2 + 2\gamma P_1 x^2 = -(1+\gamma) A_1 V_1^2 + 2\gamma P_1 x + 2\gamma A_1 V_1^2 x$$

due to $M_1^2 = \frac{A_1 V_1^2}{\gamma P_1}$, we can simplify upper one by dividing $\boxed{\gamma P_1}$

$$\rightarrow (\gamma-1) \frac{A_1 V_1^2}{\gamma P_1} x^2 + 2x^2 = -(1+\gamma)\frac{A_1 V_1^2}{\gamma P_1} + 2x + 2\gamma \frac{V_1^2 A_1}{\gamma P_1} x$$

$$\rightarrow [(\gamma-1)M^2 + 2]\, x^2 - 2(1+\gamma M^2)\, x + (1+\gamma) M^2 = 0$$

Here we got a quadratic equation with only one unknown X, according to the discrimination of roots of Unary Quadric Equation:

$$ax^2+bx+c=0$$

$$\to \quad x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Then we can get the roots of our equation:

$$X=\frac{2(1+rM_1^2)\pm\sqrt{4(1+rM_1^2)^2-4[(r-1)M_1^2+2]\times(1+r)M_1^2}}{2[(r-1)M_1^2+2]}$$

$$=\frac{(1+rM_1^2)\pm|M_1^2-1|}{2+(r-1)M_1^2}$$

Discussion: due to $M_1>1$, $|M_1^2-1|=M_1^2-1>0$

$$X_1=\frac{1+rM_1^2-(M_1^2-1)}{2+(r-1)M_1^2}=\frac{2+(r-1)M_1^2}{2+(r-1)M_1^2}=1$$

implies $\rho_1=\rho_2$, as we know if ρ_2/ρ_1 goes to 1, the shock disappears. this root should be discarded.

$$X_2=\frac{1+rM_1^2+M_1^2-1}{2+(r-1)M_1^2}=\frac{(r+1)M_1^2}{2+(r-1)M_1^2}$$

Finally,

$$\frac{\rho_2}{\rho_1}=\frac{(r+1)M_1^2}{2+(r-1)M_1^2}=\frac{r+1}{(2/M_1^2)+(r-1)}>1 \quad (6.36)$$

Higher density

When $M_1\to\infty$, $\frac{\rho_2}{\rho_1}\sim\frac{r+1}{r-1}$

∴ the medium behind a shock is always compressed

※ case 2. Pressure ↑, we would like to eliminate ρ_2 and V_2 to get the ratio of P_2 and P_1:

① → $\rho_2V_2=\rho_1V_1$ ⑤ $\qquad \rho_2=\frac{⑤^2}{⑥}=\frac{\rho_1^2V_1^2}{P_1+\rho_1V_1^2-P_2}$

② → $\rho_2V_2^2=P_1+\rho_1V_1^2-P_2$ ⑥

③ → multiply $\boxed{2(r-1)\rho_1\rho_2}$ to both sides:

$$[(r-1)\rho_1V_1^2+2rP_1]\boxed{\rho_2}=(r-1)\rho_1\boxed{\rho_2V_2^2}+2r\rho_1P_2$$

$$\to[(r-1)\rho_1V_1^2+2rP_1]\frac{\rho_1^2V_1^2}{P_1+\rho_1V_1^2-P_2}=(r-1)\rho_1(P_1+\rho_1V_1^2-P_2)+2r\rho_1P_2$$

eliminate $\rho_1P_1^2$ →

$$\left[(r-1)\frac{\rho_1V_1^2}{P_1}+2r\right]\cdot\frac{\rho_1V_1^2}{P_1}=\left[(r-1)+(r-1)\frac{\rho_1V_1^2}{P_1}+(1+r)\frac{P_2}{P_1}\right]\times\left[1+\frac{\rho_1V_1^2}{P_1}-\frac{P_2}{P_1}\right]$$

let $X=\frac{P_2}{P_1}$, $\frac{\rho_1V_1^2}{P_1}=rM_1^2$

$$[r(r-1)M_1^2+2r]\,rM_1^2=[(r-1)+r(r-1)M_1^2+(1+r)X]\times(1+rM_1^2-X)$$

$$\to[X-(rM_1^2+1)]\,[(r+1)X^2+(r-1)(1+rM_1^2)+[r(r-1)M_1^2+2r]rM_1^2]=0$$

$$\to(r+1)X^2-2(1+rM_1^2)X+(1+2rM_1^2-r)=0$$

$$\to \quad X=\frac{2(1+rM_1^2)\pm\sqrt{4(1+rM_1^2)^2-4(r+1)(1+2rM_1^2-r)}}{2(r+1)}$$

$$\therefore \quad x = \frac{(1+\gamma M_1^2) \pm |\gamma M_1^2 - \gamma|}{\gamma+1}$$

Discussion: $M_1 > 1$, then $|M_1^2 - 1| = M_1^2 - 1 > 0$

$$x_1 = \frac{1+\gamma M_1^2 + \gamma - \gamma M_1^2}{\gamma+1} = \frac{\gamma+1}{\gamma+1} = 1 \quad \text{(discard)}$$

$$x_2 = \frac{1+\gamma M_1^2 - \gamma + \gamma M_1^2}{\gamma+1} = \frac{2\gamma M_1^2 + 1 - \gamma}{\gamma+1}$$

Finally:

Higher pressure

$$\frac{P_2}{P_1} = \frac{2\gamma M_1^2 - (\gamma-1)}{\gamma+1} > 1 \quad (6.38)$$

※ case 3. Temperature ↑

Actually, after getting the change of density and pressure, we can also get the change of the temperature. As we apply the conservation of energy to the flow accross the shock wave, we can also write the basic equation ③ to $\frac{1}{2}V_1^2 + C_p T_1 = \frac{1}{2}V_2^2 + C_p T_2 = C_p T_0 = \text{Constant}$

where static temperature

$$T = \frac{P}{\rho R}$$

Higher temperature

namely
$$\frac{T_2}{T_1} = \frac{P_2}{P_1}\frac{\rho_1}{\rho_2} = \frac{[2\gamma M_1^2 - (\gamma-1)][2+(\gamma-1)M_1^2]}{(\gamma+1)^2 M_1^2}$$

Total temperature T_0 is constant, namely $\frac{T_{02}}{T_{01}} = 1$

※ Case 4 Velocity ↓

Consider conservation of momentum. the only forces acting on the control volume in the flow direction are the pressure forces, conversation of momentum applied to the control volume gives:

$$P_1 - P_2 = \rho_1 V_1 (V_2 - V_1) \quad (6.33) \Rightarrow \frac{V_2}{V_1} = \frac{\rho_1}{\rho_2} < 1$$

or

$$P_1 - P_2 = \rho_2 V_2 (V_2 - V_1)$$

Slower velocity

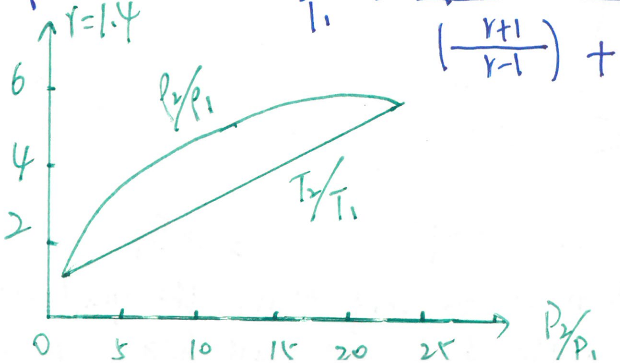
$$\rightarrow \begin{cases} V_1 V_2 - V_1^2 = \dfrac{P_1 - P_2}{\rho_1} \\ V_2^2 - V_1 V_2 = \dfrac{P_1 - P_2}{\rho_2} \end{cases}$$

Here I would like to use $\frac{P_2}{P_1}$ and γ to represent

density ratio: $$\frac{\rho_2}{\rho_1} = \frac{\left(\frac{\gamma+1}{\gamma-1}\right)\frac{P_2}{P_1} + 1}{\left(\frac{\gamma+1}{\gamma-1}\right) + \frac{P_2}{P_1}}$$

velocity ratio. $$\frac{V_2}{V_1} = \frac{\left(\frac{\gamma+1}{\gamma-1}\right) + \frac{P_2}{P_1}}{\left(\frac{\gamma+1}{\gamma-1}\right)\frac{P_2}{P_1} + 1}$$

temperature ratio: $$\frac{T_2}{T_1} = \frac{\left(\frac{\gamma+1}{\gamma-1}\right) + \frac{P_2}{P_1}}{\left(\frac{\gamma+1}{\gamma-1}\right) + \frac{P_1}{P_2}}$$



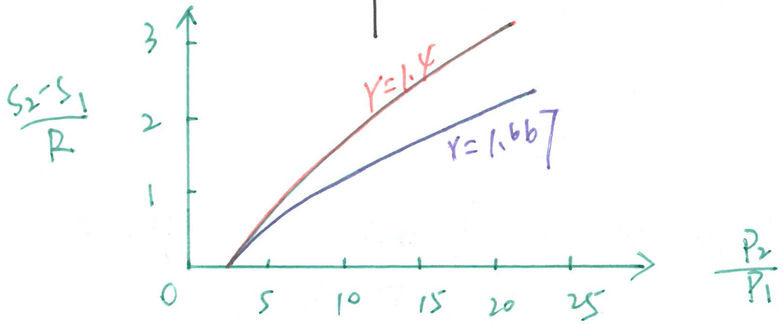
※ case 5 **Entropy** ↑

The entropy change across the shock wave is given by:

$$S_2 - S_1 = C_p \ln\left(\frac{T_2}{T_1}\right) - R\ln\left(\frac{P_2}{P_1}\right)$$

Using the relations for $\frac{T_2}{T_1}$ and $\frac{P_2}{P_1}$ given above then

$$\frac{S_2-S_1}{R} = \ln\left\{\left(\frac{P_2}{P_1}\right)^{\frac{1}{\gamma-1}}\left[\frac{(\gamma+1)\frac{P_2}{P_1}+(\gamma-1)}{(\gamma+1)+(\gamma-1)\frac{P_2}{P_1}}\right]^{-\frac{\gamma}{\gamma-1}}\right\}$$



Second law of thermodynamics:

the total entropy of an isolated system can never decrease over time.

So in our adiabatic process, the second law of thermodynamics requires the entropy must remain unchanged or must increase. namely

$$\frac{S_2-S_1}{R} \geq 0$$

$\boxed{\text{Higher Entropy}}$

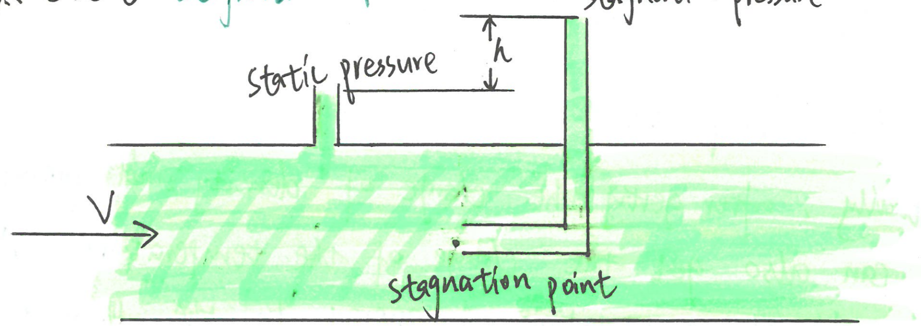
According to $\frac{P_2}{P_1} > 1$, $\gamma > 1$, $S_2 > S_1$

Because the shock wave is very thin, the gradients of velocity and temperature in the shock are very high. As a result, the effects of viscosity and heat conduction are important within the shock leading to the entropy increase across the shock wave.

$$\frac{S_2-S_1}{R} = \ln\left[\left(\frac{P_2}{P_1}\right)^{\frac{1}{\gamma-1}}\left(\frac{\rho_2}{\rho_1}\right)^{-\frac{\gamma}{\gamma-1}}\right]$$

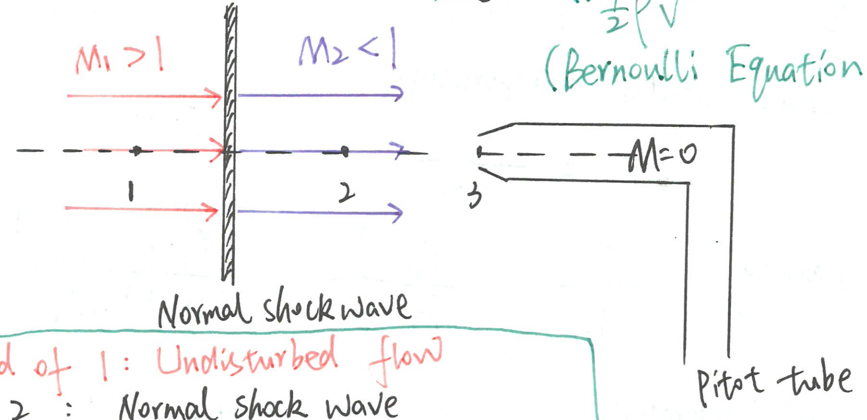
$$= \ln\left\{\left[\frac{2\gamma}{\gamma+1}(M_1^2-1)+1\right]^{\frac{1}{\gamma-1}}\left[\frac{(\gamma-1)M_1^2}{2+(\gamma-1)M_1^2}\right]^{-\frac{\gamma}{\gamma-1}}\right\}$$

※ case 6 **Stagnation Pressure**



$$P_{stag} = P_{stat} + P_{dynamic}$$

" $\frac{1}{2}\rho V^2$ (Bernoulli Equation)



Ahead of 1 : Undisturbed flow
1 → 2 : Normal shock wave
2 → 3 : Isentropic Deceleration to M=0

The stagnation pressure ratio across a normal shock wave is obtained by noting that:

$$\frac{P_0}{P} = \left(1 + \frac{\gamma-1}{2}M^2\right)^{\frac{\gamma}{\gamma-1}}$$

$$\rightarrow \frac{P_{02}}{P_{01}} = \frac{\frac{P_{02}}{P_2}}{\frac{P_{01}}{P_1}} \cdot \frac{P_2}{P_1}$$

$$= \frac{\left(1+\frac{\gamma-1}{2}M_2^2\right)^{\frac{\gamma}{\gamma-1}}}{\left(1+\frac{\gamma-1}{2}M_1^2\right)^{\frac{\gamma}{\gamma-1}}} \cdot \frac{P_2}{P_1}$$

Where $M^2 = \frac{\rho V^2}{\gamma P}$, namely

$$\frac{M_2^2}{M_1^2} = \frac{\rho_2}{\rho_1} \cdot \frac{V_2^2}{V_1^2} \cdot \frac{P_1}{P_2} = \frac{\rho_1}{\rho_2} \cdot \frac{P_1}{P_2}$$

$$\rightarrow M_2^2 = \frac{(\gamma-1)M_1^2 + 2}{2\gamma M_1^2 - (\gamma-1)}$$

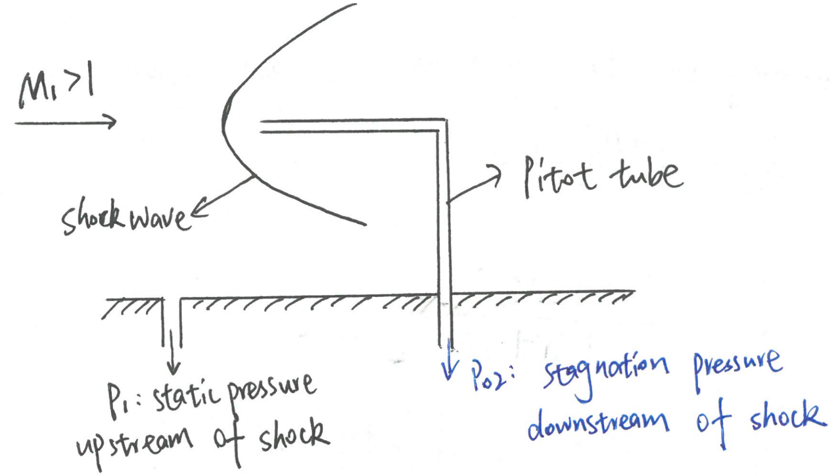
Higher stagnation Pressure

$$\therefore \frac{P_{02}}{P_{01}} = \left[\frac{(\gamma+1)M_1^2}{(\gamma-1)M_1^2+2}\right]^{\frac{\gamma}{\gamma-1}} \left[\frac{\gamma+1}{2\gamma M_1^2 - (\gamma-1)}\right]^{\frac{1}{\gamma-1}} > 1$$

$$\frac{P_{02}}{P_1} = \frac{P_{02}}{P_2} \cdot \frac{P_2}{P_1} = \left[1 + \frac{\gamma-1}{2}M_2^2\right]^{\frac{\gamma}{\gamma-1}} \cdot \left[\frac{2\gamma M_1^2 - (\gamma-1)}{\gamma+1}\right]$$

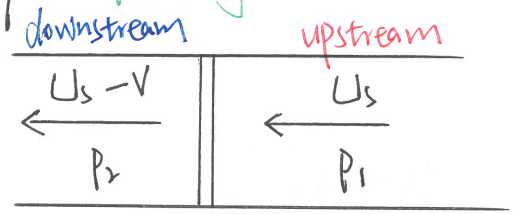
$$= \left[\frac{(\gamma+1)M_1^2}{2}\right]^{\frac{\gamma}{\gamma-1}} \cdot \left[\frac{\gamma+1}{2\gamma M_1^2 - (\gamma-1)}\right]^{\frac{1}{\gamma-1}}$$

→ Rayleigh Supersonic Pitot Tube Equation

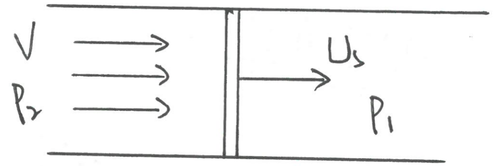


By measuring P_1 and P_{02}, we can get the Mach Number.

※ Case 7 Moving normal shock wave



① shock is fixed



② shock is moving

Then $V_1 = U_s$, $V_2 = U_s - V$ (ignore the direction)

$$M_1 = \frac{V_1}{\sqrt{\gamma P_1/\rho_1}} = \frac{U_s}{\sqrt{\gamma P_1/\rho_1}} = M_s \qquad (A_1 = \rho_1)$$

$$M_2 = \frac{V_2}{\sqrt{\gamma P_2/\rho_2}} = \frac{U_s}{\sqrt{\gamma P_2/\rho_2}} - \frac{V}{\sqrt{\gamma P_2/\rho_2}} = M_s \cdot \frac{\sqrt{\gamma P_1/\rho_1}}{\sqrt{\gamma P_2/\rho_2}} - M_2'$$

where $M_2' = \frac{V}{\ldots}$

According to the equations we got above:

$$\frac{P_2}{P_1} = \frac{2\gamma M_s^2 - (\gamma-1)}{\gamma+1}$$

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1) M_s^2}{2 + (\gamma-1) M_s^2}$$

$$\frac{T_2}{T_1} = \frac{\left[2\gamma M_s^2 - (\gamma-1)\right]\left[2 + (\gamma-1) M_s^2\right]}{(\gamma+1)^2 M_s^2}$$

$$M'^2 = \frac{(\gamma-1) M_s^2 + 2}{2\gamma M_s^2 - (\gamma-1)}$$

Summary:

Density: $$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1) M_1^2}{2 + (\gamma-1) M_1^2} \quad ①$$

Pressure: $$\frac{P_2}{P_1} = \frac{2\gamma M_1^2 - (\gamma-1)}{\gamma+1} \quad ②$$

Temperature: $$\frac{T_2}{T_1} = \frac{\left[2\gamma M_1^2 - (\gamma-1)\right]\left[2 + (\gamma-1) M_1^2\right]}{(\gamma+1)^2 M_1^2} \quad ③$$

Velocity: $$\frac{V_2}{V_1} = \frac{\rho_1}{\rho_2} = \frac{2 + (\gamma-1) M_1^2}{(\gamma+1) M_1^2} \quad ④$$