Section 7.4 -7
Linear theory of waves & instability

Yuta Yamazaki
NAOJ / Univ. Tokyo (M1)

Basic Seminar IIA

7.4 Perturbations at a two-fluid interface

Irrotational Flow of **barotropic ideal** fluid with **conservative force**

- **”Irrotational” …** **$\Gamma \equiv \nabla \times \mathbf{v} = 0$**
  - **v** can be written as the gradient of a certain scalar potential

    $\mathbf{v} = \nabla \phi$ ← we can use ϕ instead of v

Kelvin’s vorticity theorem

- for barotropic ideal fluid with conservative body force

  $D\Gamma/Dt = 0$

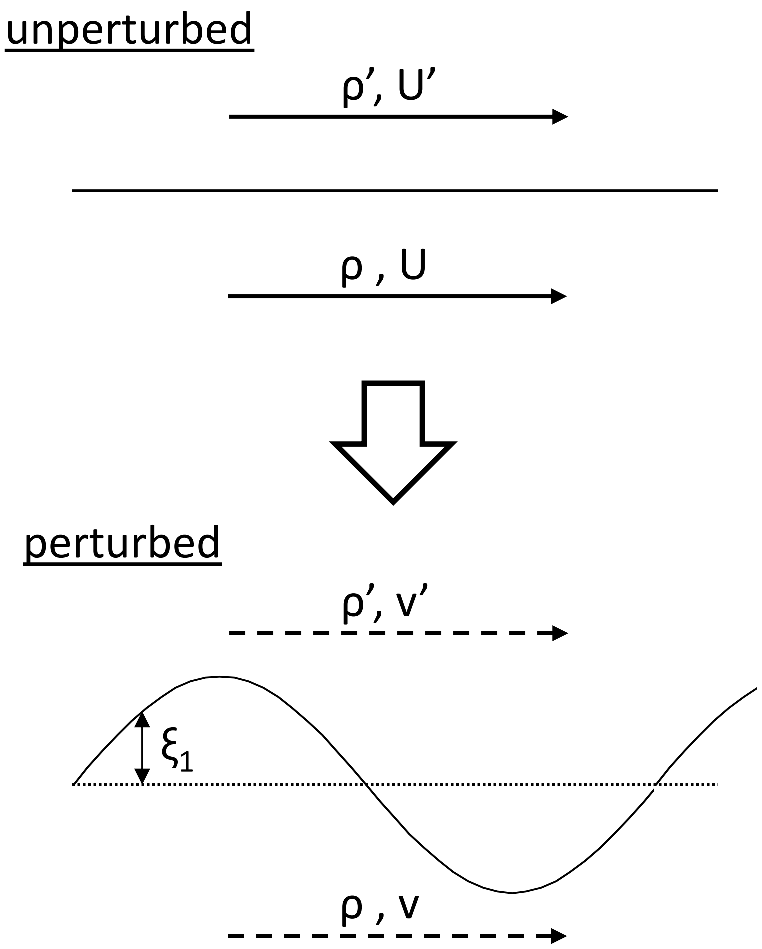
- **Euler eq.** (**Navier-Stokes eq. for ideal**(=inviscid) **fluid**) under conservative force

$$-\frac{\partial \phi}{\partial t} + \frac{1}{2}v^2 + \frac{p}{\rho} + \Phi = \boxed{F(t)} \quad \dots (7.29)$$

does not depend on position

7.4 Perturbations at a two-fluid interface

- **Assumptions**
  - **Incompressible** (⇔ $\mathbf{\nabla \cdot v = 0}$ ⇔ continuity eq.)
    - each fluids have constant densities: ρ and ρ' (which do not vary with position.)
  - **Irrotational** ($\mathbf{\Gamma \equiv \nabla \times v = 0}$)
    - $\mathbf{v = -\nabla \phi}$
  - **Conservative force** ($\mathbf{F} = \nabla \boldsymbol{\Phi}$)
    - Note: Be sure to distinguish between ϕ and Φ !!!



7.4 Perturbations at a two-fluid interface

- **formulation**
  - general config.

velocity in lower/upper fluid:

$$\begin{aligned} v(x,z,t) &= \underset{\text{unperturbed}}{\boxed{\begin{matrix} U \\ U' \end{matrix}}} \end{aligned}$$

$$\begin{aligned} v(x,z,t) &= U + v_1 , \\ v'(x,z,t) &= U' + v'_1 . \end{aligned}$$

unperturbed perturbed

$\Rightarrow$

velocity potentials:

$$\begin{aligned} \phi &= -U\ x + \phi_1 \\ \phi' &= -U'\ x + \phi'_1 . \end{aligned}$$

unperturbed perturbed

  - boundary conditions

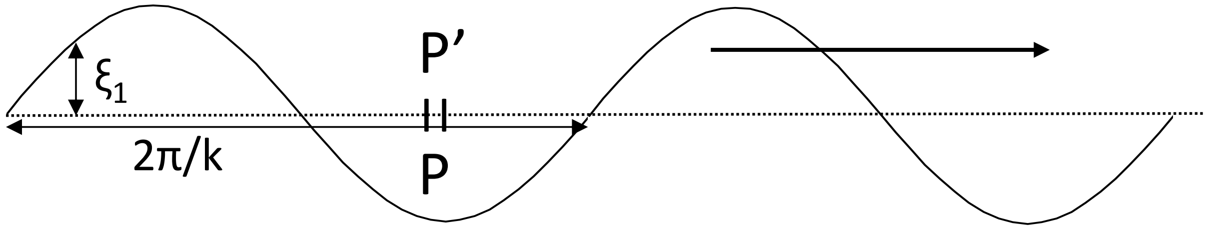
1. the perturbations will vanish @ far away from the interface $\Rightarrow$ $\phi_1, \phi'_1 \rightarrow 0 \ (@\ z \rightarrow \infty \ / \ -\infty)$
2. Pressure has to be continuous across the interface $\Rightarrow$ $P(z=0) = P'(z=0)$

7.4 Perturbations at a two-fluid interface

- **formulation**

$\phi'_1 \rightarrow 0$

ρ' , ϕ'



ρ , ϕ

$\phi_1 \rightarrow 0$

periodic symmetry along x-axis $\rightarrow \xi_1 = f(t) \cdot \exp(ikx)$

$\swarrow$ **FT**

$$\xi_1(\omega) = A \exp(-i\omega t) \cdot \exp(ikx) \quad \dots (7.36)$$

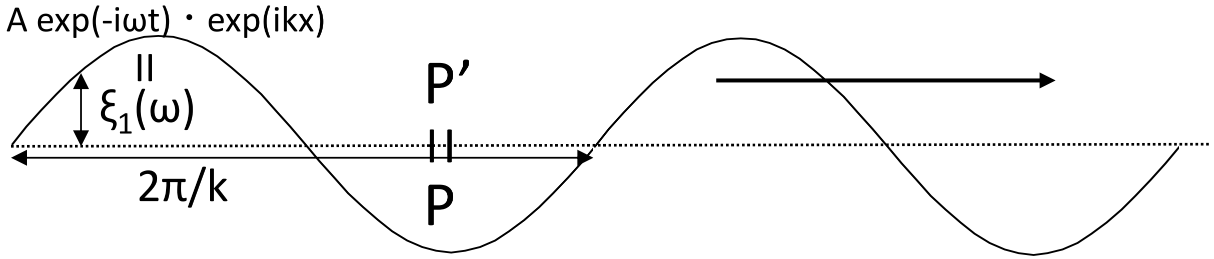
if we can linearize the perturbation eq., <u>each mode evolve separately</u>.
Then we can consider one mode as a representative.

7.4 Perturbations at a two-fluid interface

- **formulation**

$\phi'_1 \rightarrow 0$

ρ' , ϕ'



ρ , ϕ

$\phi_1 \rightarrow 0$

Linearized Perturbation eq.

vertical velocity at vicinity of z=0

$$-\frac{\partial \phi_1}{\partial z} = \frac{\partial \xi_1}{\partial t} + U\frac{\partial \xi_1}{\partial x} \qquad \text{… (7.34,35)}$$

Poison eq.

Incompressive & Irrotational

$$\rightarrow \nabla^2 \phi = 0 \qquad \text{… (7.31)}$$

$$\phi_1 = C \exp(-i\omega t) \cdot \exp(ikx + kz) \qquad \text{… (7.37)}$$

$$\phi'_1 = C' \exp(-i\omega t) \cdot \exp(ikx - kz) \qquad \text{… (7.38)}$$

7.4 Perturbations at a two-fluid interface

- **Boundary condition at the interface**

from (7.29) : $-\frac{\partial \phi}{\partial t} + \frac{1}{2}v^2 + \frac{p}{\rho} + \Phi = F(t)$

we can calculate pressure **P** & **P'** at z=0

and obtain (7.45) : $\rho\left(-\frac{\partial \phi_1}{\partial t} - U\frac{\partial \phi_1}{\partial x} + g\xi_1\right) = \rho'\left(-\frac{\partial \phi'_1}{\partial t} - U'\frac{\partial \phi'_1}{\partial x} + g\xi_1\right)$ at vicinity of z=0

… (7.45)

substituting ξ_1 , ϕ_1 , ϕ'_1 into

ξ-ϕ relation eq. (= 7.34, 35) and this boundary condition (7.45), finally we get following **dispersion relation**.

$$\frac{\omega}{k} = \frac{\rho U + \rho' U'}{\rho + \rho'} \pm \left(\frac{g}{k}\frac{\rho - \rho'}{\rho + \rho'} - \frac{\rho\rho'(U - U')^2}{(\rho + \rho')^2}\right)^{1/2} \quad \text{… (7.48)}$$

7.4 Perturbations at a two-fluid interface

- **Summary of Linear Analysis for perturbationa at a two-fluid interface**
  - perturbed interface

    … ξ_1 = A exp(-iωt)・exp(ikx)

  - dispersion relation

    … $$\frac{\omega}{k} = \frac{\rho U + \rho' U'}{\rho + \rho'} \pm \left(\boxed{\frac{g}{k}\frac{\rho - \rho'}{\rho + \rho'} - \frac{\rho \rho' (U - U')^2}{(\rho + \rho')^2}} \right)^{1/2}$$

    ≡ D

    D>0 ⟶ perturbations will not grow

    ex.) ρ>ρ’ , U=U’=0 … **surface gravity wave**

    D<0 ⟶ perturbations will grow

    ex1.) ρ<ρ’ … **Rayleigh-Taylor instability**

    ex2.) ρ>ρ’ … **Kelvin-Helmholtz instability**

7.4 Perturbations at a two-fluid interface

surface gravity wave

ρ' $U' = 0$

$\wedge$

ρ $U = 0$

Rayleigh-Taylor instability

ρ'

$\vee$

ρ

Kelvin-Helmholtz instability

$$k > \frac{(\rho^2 - \rho'^2)g}{\rho\rho'(U - U')^2}$$

ρ'

$\wedge$

ρ

7.5 Jeans Instability

perturbation

$\rho : \rho_0 \rightarrow \rho_0 + \rho_1$

- **Let us analyze instability of a uniform gas bound by self-gravity**
  - Poisson eq. for gravity … (7.54, 57)
  - continuity eq. … (7.55)
  - Euler eq. + hydrostatic eq. … (7.56)

(using to eliminate **P)**

let all variables have following x and t dependences:

exp[i (kx-ωt)]

then we obtain dispersion relation … (7.58, 59)

$$\omega^2 = c_s{}^2(k^2 - k_J{}^2) \quad , \qquad k_j{}^2 = \frac{4\pi G\rho_0}{c_s{}^2}$$

Jeans Mass

$M_J = 4/3\ \pi\ \lambda_J{}^3\rho_0$

perturbations will grow if the size of them is lager than $\lambda_J \equiv 2\pi/k_J$!

7.6 Stellar Oscillations. Helioseismology

- **Let us analyze radial pulsation of stars**

from Euler eq., we obtain the acceleration of a fluid element … (7.61)

$$\frac{dv}{dt} = -\frac{1}{\rho}\frac{dp}{dr} - \frac{GM}{r^2}$$

assuming uniform expansion and considering only linear term,

$d\rho/dr = -3\ \rho_0/r_0$, $dp/dr = \gamma\ (d\rho/dr)\ p_0/\rho_0 = -3\gamma\ p_0/r_0$

substituting these to (7.61) and eliminate p by using hydrostatic eq.,
we obtain (7.64)

$$\frac{dv}{dt} = -\frac{GM}{{r_0}^2} 3\delta \left(\gamma - \frac{4}{3}\right)$$

in the case of $\delta>0$, when $\gamma>4/3$ acceleration become negative, so that stars are stable.