7 Linear theory of waves and instabilities

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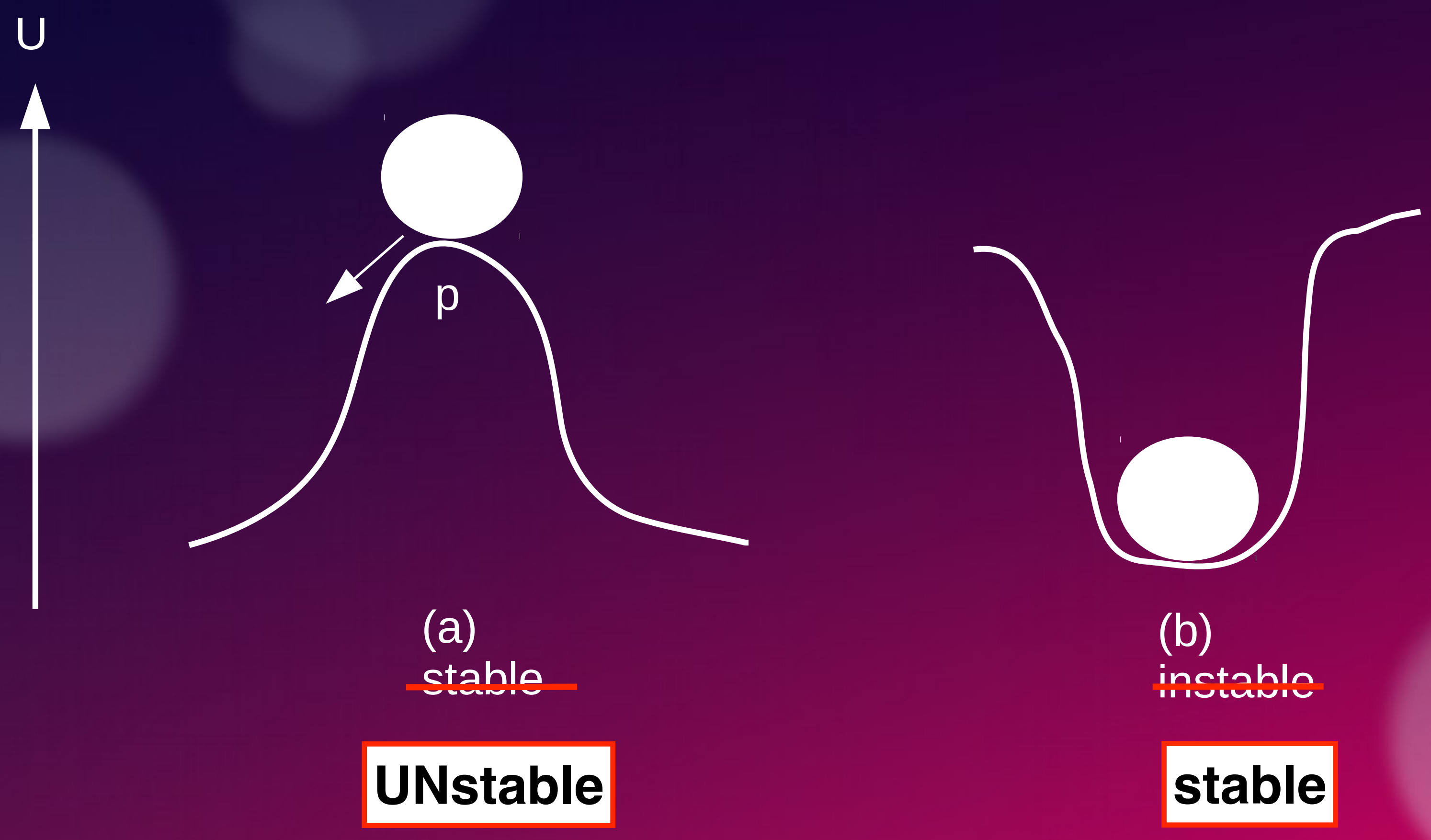
7.1 The philosophy of perturbation analysis



Hydrodynamic case

Steady state ∂/∂t=0 really means stable??

=> No.

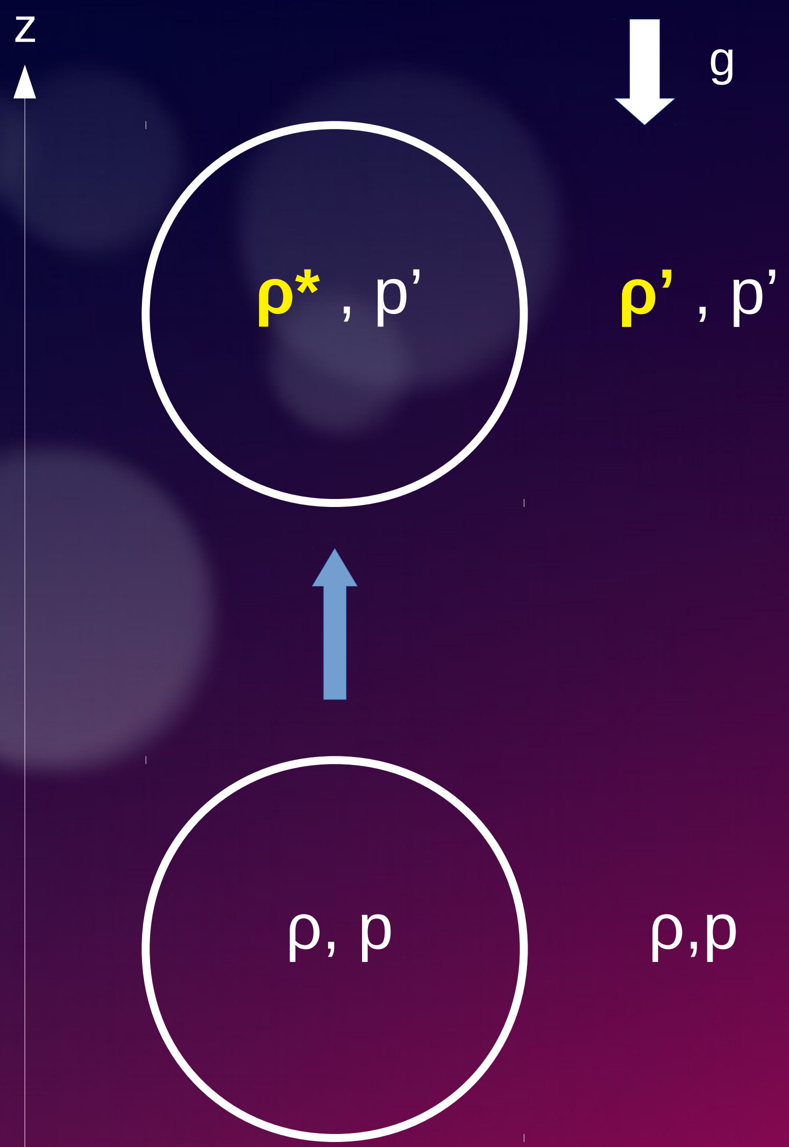
=> Growth of **perturbations δ(t)**

Perturbations are so small => **Linearization**

- principal of superposition
- Evolution is separated in isolated perturbation

*The linear theory just tell us
how the perturbation will initially grow

7.2 Convective instability and internal gravity waves



Blob of gas

- time to be pressure equilibrium
  t_p ~ (system size)/ Cs

- time to be thermal equilibrium
  t_th ~ transportation time of energy

=> **t_p << t_th ,** blob is **adiabatic**

ρ* < ρ' => buoyant and continue to go upward
=> unstable

ρ* > ρ' => return to original position
=> stable

In adiabatic process,

$$\rho^* = \rho(p'/p)^{(1/\gamma)} \quad \ldots (7.1)$$

For the pressure,

$$p' = p + (dp/dz)\Delta z \ \ldots \Rightarrow (7.1) \Rightarrow \text{linear order in } \Delta z$$

$$\Rightarrow \rho^* = \rho + \rho/\gamma p\,(dp/dz)\Delta z \quad \ldots (7.2)$$

Blob of gas

For the density,

$$\rho' = \rho + (d\rho/dz)\Delta z$$

Using $\rho = p/RT$ (eos),

$$\Rightarrow \rho' = \rho + \rho/p(dp/dz)\Delta z - \rho/T(dT/dz)\Delta z \quad \ldots (7.3)$$

atmosphere

From (7.2)&(7.3),

$$\rho^* - \rho' = [-(1-1/\gamma)\rho/p(dp/dz) + \rho/T(dT/dz)]\Delta z \quad \ldots (7.4)$$

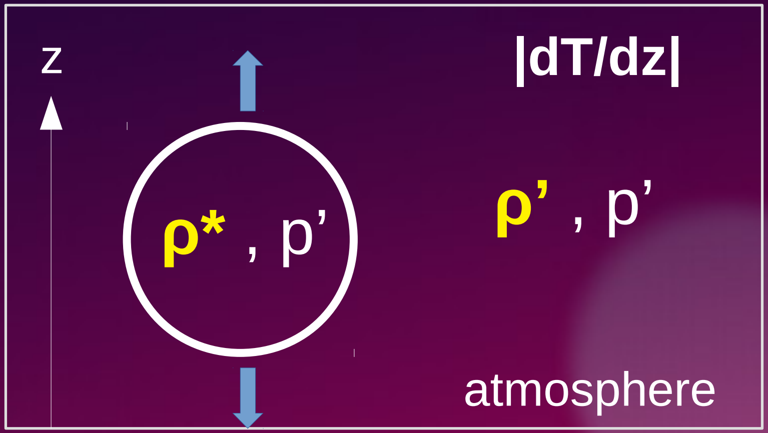
negative

$\rho^* > \rho'$ => stable

=> $|dT/dz| < (1-1/\gamma)T/p|dp/dz|$ ...(7.5)

=> **<u>Schwarzschild stability condition</u>**

Temperature gradient **$|dT/dz|$** is important for whether stable or not.



EOM of displaced blob is…

$\rho^*(d^2/dz^2)\Delta z = -(\rho^*-\rho')g$ => (7.4)

=> $\mathbf{(d^2/dz^2)\Delta z + N^2\,\Delta z = 0}$ ...(7.6)

where $\mathbf{N = g/T\,[(dT/dz)-(1-1/\gamma)T/\rho(dp/dz)]^{1/2}}$ ...(7.7)

Stable condition (7.5) => N is **real** number

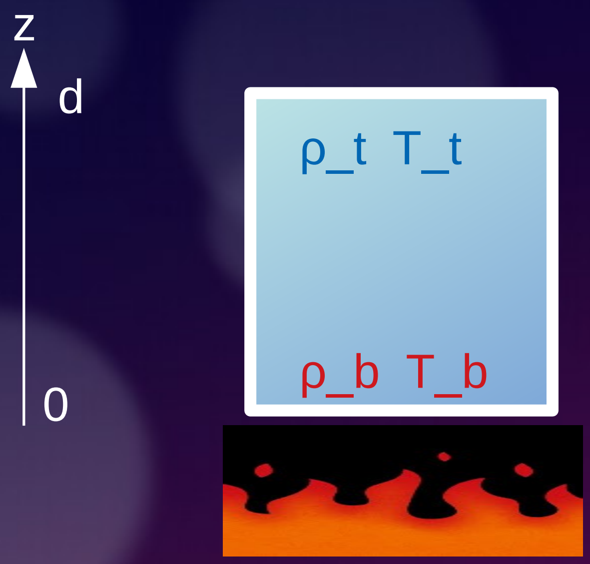
=> solution of Eq(7.6) represents **oscillatory** motion

=> **Internal gravity waves**

*This is **not** full perturbation analysis.

7.3 Rayleigh-Be'nard convection

Nearly incompressible liquid : water



$\rho_b < \rho_t$ => come on top of the colder liquid

$d\varepsilon = c_p\, dT$

$\rho(\partial\varepsilon/\partial t + \mathbf{v} \cdot \nabla\varepsilon) - \nabla \cdot (K\nabla T) + p\nabla \cdot \mathbf{v} = 0$ ...(3.53)

=> $\partial T/\partial t + \mathbf{v} \cdot \nabla T = \kappa\nabla^2 T$ ...(7.8)

K : thermal conductivity (const)
$\kappa = K/(\rho\, c_p)$: thermometric conductivity

(7.8) <= $\partial/\partial t = 0, \mathbf{v} = 0$ => $T_0(z) = T_b - \boldsymbol{\beta} z$ ...(7.10) $\boldsymbol{\beta} = (T_b - T_t)/d$

=> $\rho_0(z) = \rho_b(1 + \alpha\beta z)$ ...(7.11)

α : coefficient of volume expantipon

Perturbations around the equilibrium state

$T_0 \;\Rightarrow\; T_0$ **+T_1**
$\rho_0 \;\Rightarrow\; \rho_0$ **– $\rho_b\, \alpha\, T_1$**
$p_0 \;\Rightarrow\; p_0$ **+p_1**

V1 : arising out of perturbation

Equilibrium state => $dp_0/dz=-\rho_0(z)g$ ...(7.12)

Perturbation of Navier-Stokes equation

$$(\rho_0 - \rho_b\, \alpha T_1)[\partial \mathbf{v1}/\partial t+(\mathbf{v1} \cdot \nabla)\mathbf{v1}] = -\nabla(p_0+p1)+(\rho_0 - \rho_b\, \alpha T_1)\mathbf{g}+\mu\nabla^2 \mathbf{v1} \quad \text{… (7.13)}$$

$$\rho_b(\partial \mathbf{v1}/\partial t) = -\nabla p1 - \rho_b\, \alpha T_1 \mathbf{g}+\mu\nabla^2 \mathbf{v1} \quad \text{...(7.14)}$$

(7.8),(7.10) => $\partial(\cancel{T_0}+T_1)/\partial t + \mathbf{v} \cdot \nabla(\cancel{T_0}+T_1) = \kappa\nabla^2 (\cancel{T_0}+T_1)$

$$\Rightarrow \boxed{\partial T_1/\partial t = \beta v_{1z} + \kappa\nabla^2 T_1 \quad \text{...(7.15)}}$$

Curl of (7.14) twice,

$$\partial/\partial t\ \nabla^2 v_{1z} = \alpha g(\partial^2 T_1/\partial x^2 + \partial^2 T_2/\partial y^2) + \nu\nabla^4 v_{1z} \quad ...(7.16)$$

$$\partial T_1/\partial t = \beta v_{1z} + \kappa\nabla^2 T_1 \quad ...(7.15)$$

- 2 equations and 2 variables (T_1, v_{1z})
- linear
- arbitrary perturbation as a superposition of Fourier component

$$v_{1z}(z,t) = W(z) \exp(\sigma t + ik_x x + ik_y y) \quad ...(7.17)$$

$$T_1(z,t) = \theta(z) \exp(\sigma t + ik_x x + ik_y y) \quad ...(7.18)$$

- $\sigma > 0$: perturbation grow
- $\sigma < 0$: decaying perturbation
- $\sigma = 0$: ??? (marginally stable)

(7.17-7.18), $\sigma=0$ => (7.15,7.16)

$$\beta W + \kappa(d^2/dz^2 - k^2)\,\theta = 0 \quad \dots (7.19)$$

$$-\alpha g k^2\,\theta + \nu(d^2/dz^2 - k^2)^2 W = 0 \quad \dots (7.20)$$

$k=k_x^2+k_y^2$. On eliminating θ, and we get

$$\nu\kappa(d^2/dz^2 - k^2)^3 W = -\alpha\beta g k^2 W \quad \dots (7.21)$$

$z=z'd$, $k=k'/d$

$$(d^2/dz'^2 - k'^2)^3 W = -R k'^2 W \quad \dots (7.22)$$

$R=\alpha\beta g d^4/\kappa\nu$: **Rayleigh number**

Sixth-order differential => we need six boundary condition

The simplest case

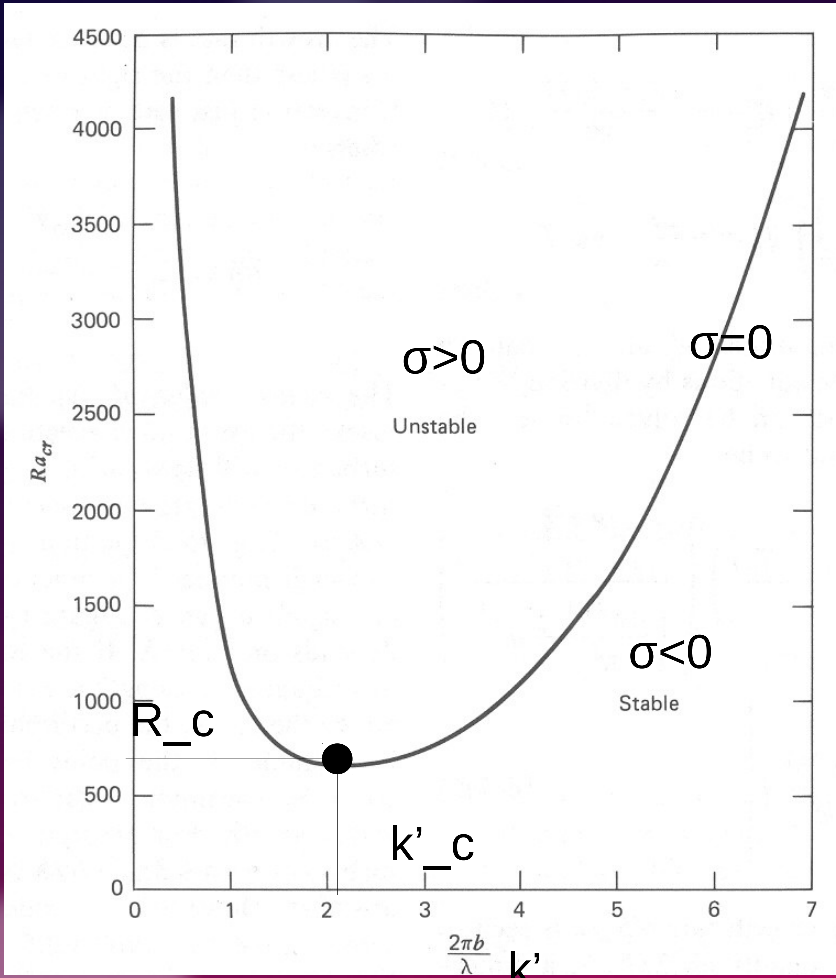
$\rho^* > \rho'$ => **stable**

=> **$|dT/dz|$ <** $(1-1/\gamma)T/p|dp/dz|$ ...(7.5)

$W(z)=W_0 \sin\pi z'$ … (7.24)

=> (7.22) => $R = (\pi^2+k'^2)^3 / k'^2$ ...(7.25)

($R=\alpha\beta g d^4/\kappa\nu \propto \beta =$ **$|dT/dz|$** ...(7.23))



$k'_c =$ **$(\pi^2 / 2)^{1/2}$**

$R_c = 27\pi^4 / 4 =$ **657.5**

- $R<R_c$:
system is **stable** against all perturbations

-$R>R_c$:
System is **unstable** to **arbitrary** perturbations