2018/10/05

Basic seminar II A

-+Reviews of Chap. 2-5+-

The physics of fluids and plasmas
-An introduction for astrophysicists-

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The full set of hydrodynamic equation for neutral fluids

- The continuity equation (conservation of mass)

$$\frac{\partial \rho}{\partial \mathrm{t}} + \nabla \cdot (\rho \mathbf{v}) = 0$$

- Incompressible Navier-Stokes equation (conservation of momentum)

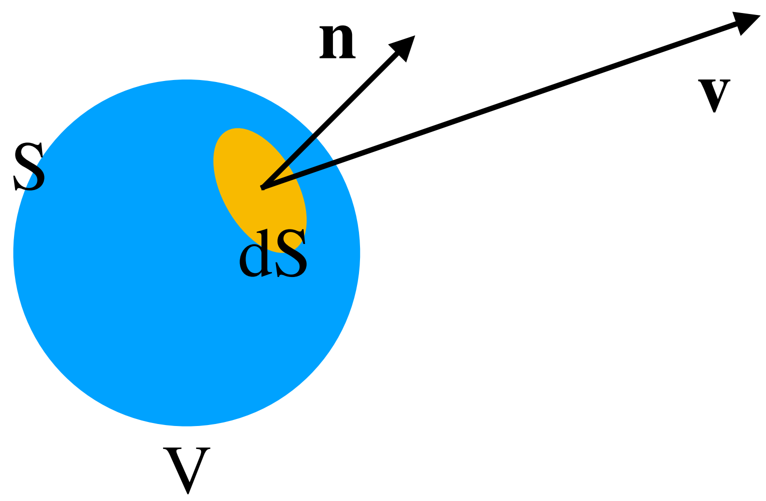
$$\frac{\partial \mathbf{v}}{\partial \mathrm{t}} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho}\nabla \mathrm{p} + \mathbf{F} + \nu \nabla^{\mathbf{2}}\mathbf{v}$$

- Conservation of energy

$$\rho\left(\frac{\partial \epsilon}{\partial \mathrm{t}} + \mathbf{v} \cdot \nabla \epsilon\right) - \nabla \cdot (\mathrm{K} \nabla \mathrm{T}) + \mathrm{p} \nabla \mathbf{v} = 0$$

The continuity equation (conservation of mass)

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0$$



total mass

$$\iiint_V \rho\, dV$$

increase of mass per unit time

$$\frac{\partial}{\partial t}\iiint_V \rho\, dV = \iiint_V \frac{\partial\rho}{\partial t} dV$$

the mass flows out from the region V

$$\iint_S \rho\mathbf{v}\cdot\mathbf{n}\, dS = \iiint_{\mathbf{V}} \nabla\cdot(\rho\mathbf{v})\, dV$$

conservation of the mass

$$\iiint_V \frac{\partial\rho}{\partial t} dV = -\iiint_V \nabla\cdot(\rho\mathbf{v})\, dV$$

$$\iiint_V \left(\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v})\right) dV = 0$$

So that this makes ends meet
for any arbitrary regions V

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0$$

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Incompressible Navier-Stokes equation (conservation of momentum)

$$\nabla \cdot \mathbf{v} = 0 \rightarrow \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \rightarrow \frac{\partial \rho}{\partial t} = 0 \; , \rho = \text{const}.$$

The continuity equation

internal source

external source

(force per unit mass)

$$\frac{d\mathbf{v}}{dt} = \frac{\mathbf{F}}{m}$$

equation of motion

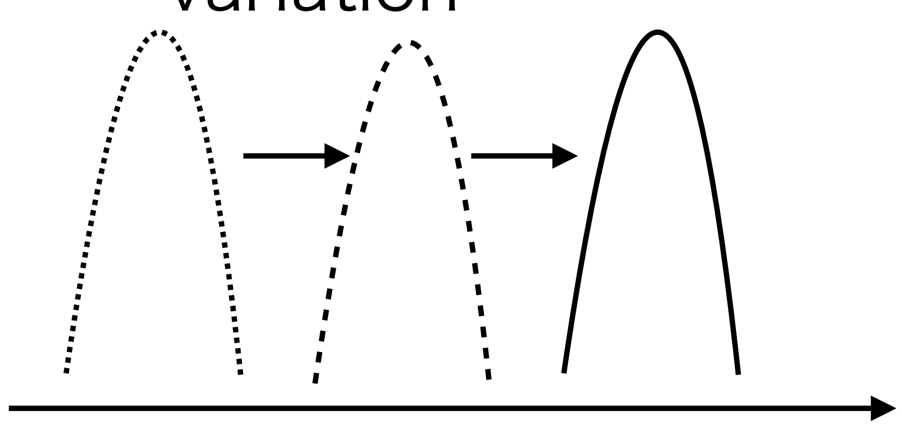
convection

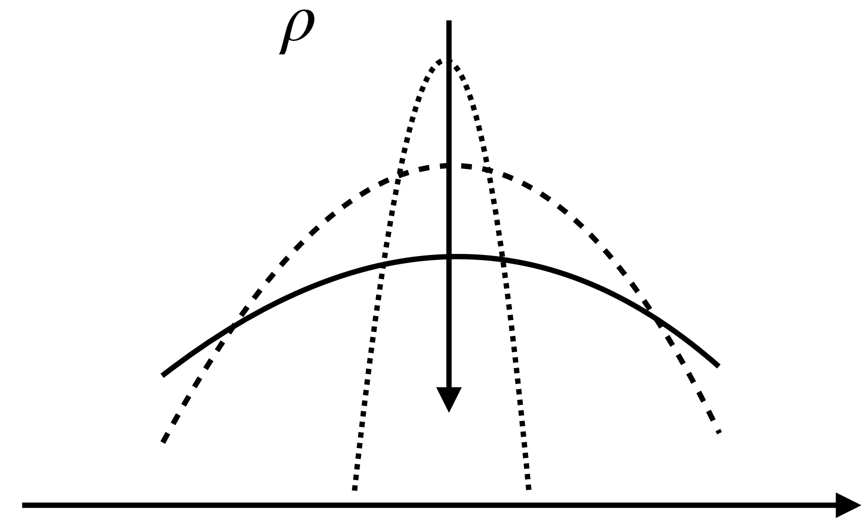
$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p + \mathbf{F} + \nu \nabla^2 \mathbf{v}$$

diffusion

kinematic viscosity : $\nu = \frac{\mu}{\rho}$ μ : viscosity coefficient

variation





The full set of hydrodynamic equation for neutral fluids

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conservation of energy

going in and out of energy
with fluids

going in and out of energy
due to the **work**

$$\rho\left(\frac{\partial \epsilon}{\partial t} + \mathbf{v} \cdot \nabla \epsilon\right) - \nabla \cdot (\mathrm{K} \nabla \mathrm{T}) + \mathrm{p} \nabla \mathbf{v} = 0$$

going in and out of energy
due to the **heat flow**

The full set of hydrodynamic equation for neutral fluids

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- Conservation of energy

$$\rho\left(\frac{\partial \epsilon}{\partial \mathrm{t}} + \mathbf{v} \cdot \nabla \epsilon\right) - \nabla \cdot (\mathrm{K} \nabla \mathrm{T}) + \mathrm{p} \nabla \mathbf{v} = 0$$

Assumptions

1. Spatial variation of viscosity μ is neglected.

2. Heat production due to the viscous damping of motion is neglected.

3. Neutral incompressible fluids $\nabla \cdot \mathbf{v} = 0$

No radiation **No chemical evolution**

- The continuity equation $$\frac{\partial \rho}{\partial \mathrm{t}} + \nabla \cdot (\rho \mathbf{v}) = 0$$
- Incompressible Navier-Stokes equation $$\frac{\partial \mathbf{v}}{\partial \mathrm{t}} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho}\nabla \mathrm{p} + \mathbf{F} + \nu \nabla^2 \mathbf{v}$$
- Conservation of energy $$\rho\left(\frac{\partial \epsilon}{\partial \mathrm{t}} + \mathbf{v} \cdot \nabla \epsilon\right) - \nabla \cdot (\mathrm{K} \nabla \mathrm{T}) + \mathrm{p} \nabla \mathbf{v} = 0$$

We will consider the solutions of the equations under different circumstance.

Ideal fluids

Chapter 4, no viscosity

Viscous flows

Chapter 5

Euler equation

Incompressible Navier-Stokes equation $\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p + \mathbf{F} + \nu \nabla^2 \mathbf{v}$

If $\nu = 0$ $(\mu = 0)$, $\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p + \mathbf{F}$ **Euler equation**

no viscosity

&

$$(\mathbf{v} \cdot \nabla)\mathbf{v} = \frac{1}{2}\nabla(\mathbf{v} \cdot \mathbf{v}) - \mathbf{v} \times (\nabla \times \mathbf{v})$$

$$\longrightarrow \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} + \frac{1}{2}\nabla(\mathbf{v} \cdot \mathbf{v}) - \mathbf{v} \times (\nabla \times \mathbf{v}) = -\frac{1}{\rho}\nabla p + \mathbf{F}$$

take a curl

- **F** is conservative force
- vorticity $\omega = \nabla \times \mathbf{v}$
- incompressible fluids $\nabla \cdot \mathbf{v} = 0$ $\longrightarrow$ $\rho = \text{const.}$

$$\frac{\partial \omega}{\partial t} = \nabla \times (\mathbf{v} \times \omega)$$

the complete dynamical theory of incompressible fluids

Newtonian fluid

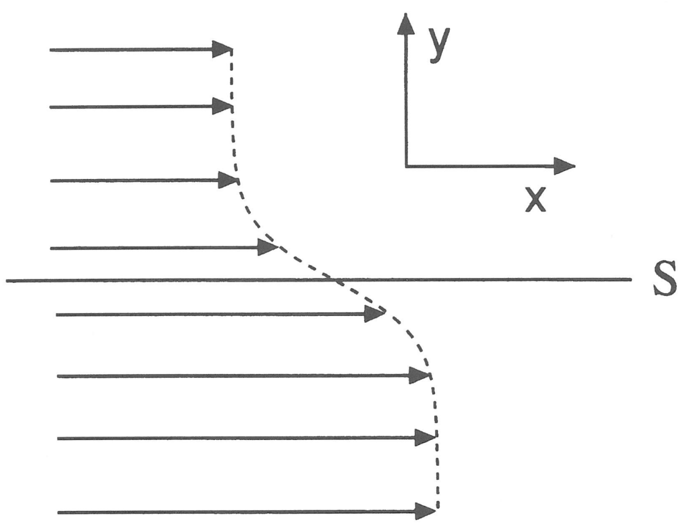
Shear force is proportional to the gradient of velocity

Force
- acts in a **vertical** direction on the surface of fluids → pressure
- acts in a **horizontal** direction on the surface of fluids → shear force

Tensor $P_{ij} = p\,\delta_{ij} + \pi_{ij}$

$$\pi_{ij} = a\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) + b\,\delta_{ij}\nabla\cdot\mathbf{v}$$

Figure 4.1 A fluid with two layers moving with different velocities.



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The equation of motion $\rho\frac{dv_i}{dt} = \rho F_i - \frac{\partial P_{ij}}{\partial x_j}$

$$\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p + \mathbf{F} + \nu\nabla^2\mathbf{v}$$

Navier-Stokes equation

take a curl →

$$\frac{\partial\omega}{\partial t} = \nabla\times(\mathbf{v}\times\omega) + \nu\nabla^2\omega$$

Reynolds number

Fluid flows around geometrically similar object of different sizes

let L, V are the typical length of the object and fluid velocity

$$\mathbf{x} = \mathbf{x}'L, \quad \mathbf{v} = \mathbf{v}'V, \quad t = t'\frac{L}{V}, \quad \omega = \omega'\frac{V}{L}$$

$\boldsymbol{x}'$, $\boldsymbol{v}'$, t' and ω' are the values of length, velocity, time and vorticity measured in these scaled units

$$\frac{\partial \omega}{\partial t} = \nabla \times (\mathbf{v} \times \omega) + \nu \nabla^2 \omega \longrightarrow \frac{\partial \omega'}{\partial t'} = \nabla \times (\mathbf{v}' \times \omega') + \frac{\mathbf{1}}{\mathscr{R}} \nabla'^2 \omega'$$

the equation is in the scaled variables

Reynolds number $\mathscr{R} = \dfrac{LV}{\nu}$

Figure 5.3 Flow of a viscous fluid past a cylinder for various Reynolds numbers. Reproduced from Batchelor (1967). Originally taken from Homann (1936).